

# SHARP $L^p$ ESTIMATES ON BMO

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**ABSTRACT.** We construct the upper and lower Bellman functions for the  $L^p$  (quasi)-norms of BMO functions. These appear as solutions to a series of Monge–Ampère boundary value problems on a non-convex plane domain. The knowledge of the Bellman functions leads to sharp constants in inequalities relating average oscillations of BMO functions and various BMO norms.

## 1. INTRODUCTION

For a measurable set  $E \subset \mathbb{R}^n$  with finite, non-zero Lebesgue measure  $|E|$ , and a locally integrable real-valued function  $\varphi$ , let  $\langle \varphi \rangle_E$  denote the average of  $\varphi$  over  $E$ ,

$$\langle \varphi \rangle_E = \frac{1}{|E|} \int_E \varphi.$$

For a function  $\varphi$ , a cube  $Q$ , and  $p > 0$ , let  $\|\varphi\|_{L^p(Q)}$  denote the  $Q$ -normalized  $L^p$  (quasi)-norm of  $\varphi$ ,

$$\|\varphi\|_{L^p(Q)} = \langle |\varphi|^p \rangle_Q^{1/p}.$$

Observe that  $\|\varphi\|_{L^p(Q)}$  is increasing in  $p$ :

$$(1.1) \quad \|\varphi\|_{L^{p_1}(Q)} \leq \|\varphi\|_{L^{p_2}(Q)}$$

for  $p_1 \leq p_2$ , with equality happening if and only if  $\varphi = \text{const}$  on  $Q$ .

Fix a cube  $Q$  in  $\mathbb{R}^n$  and let  $\text{BMO}^p(Q)$  be the (factor-)space

$$(1.2) \quad \text{BMO}^p(Q) = \{\varphi \in L^1(Q) : \langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J \leq C^p < \infty, \forall \text{ cube } J \subset Q\},$$

with the smallest such  $C$  being the corresponding norm (quasi-norm for  $0 < p < 1$ ),

$$(1.3) \quad \|\varphi\|_{\text{BMO}^p(Q)} = \sup_{\text{cube } J \subset Q} \langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J^{1/p}.$$

It is known that all  $p$ -based norms defined by (1.3) are equivalent and so (1.2) defines the same space for all  $p > 0$ . This fact is usually seen as a consequence of the John–Nirenberg inequality, although using that inequality to prove it will produce suboptimal constants of norm equivalence. One of the primary motivations of this work is to quantify this equivalence precisely, in dimension 1. To this end, we relate all  $\text{BMO}^p$  norms to the  $\text{BMO}^2$  norm. The reason  $\text{BMO}^2$  norm plays a central role here is that it allows us to take advantage of the self-duality of  $L^2(Q)$ . From now on, we reserve the name BMO for  $\text{BMO}^2$ :

$$\text{BMO}(Q) = \{\varphi \in L^1(Q) : \langle \varphi^2 \rangle_J - \langle \varphi \rangle_J^2 \leq C^2 < \infty, \forall \text{ cube } J \subset Q\}.$$

Thus, we would like to find the best constants  $c_p, C_p$  in the double inequalities

$$(1.4) \quad c_p \|\varphi\|_{\text{BMO}(Q)} \leq \sup_{\text{interval } J \subset Q} \langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J^{1/p} \leq C_p \|\varphi\|_{\text{BMO}(Q)}, \quad p > 0.$$

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Some cases are trivial: (1.1) implies that  $C_p = 1$  for  $p \leq 2$ , and  $c_p = 1$  for  $p \geq 2$  (the equalities hold, for instance, for any  $\varphi$  with zero average and constant modulus on  $Q$ ).

To find  $c_p$  and  $C_p$ , we estimate, for any  $\varphi \in \text{BMO}(Q)$ , the quantity  $\langle |\varphi - \langle \varphi \rangle_Q|^p \rangle_Q$  (we will refer to this as the  $p$ -oscillation of  $\varphi$  over  $Q$ ) in terms of its 2-oscillation  $\langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2$  and  $\|\varphi\|_{\text{BMO}(Q)}$ . Our estimates are sharp for any  $p > 0$  and one can use them to estimate the  $p$ -oscillation of a function in terms of its  $q$ -oscillation and BMO norm, for any  $0 < p, q < \infty$ . What is surprising is that, due to the nature of the optimizers in the  $p \leftrightarrow 2$  inequalities, we obtain *sharp*  $p \leftrightarrow q$  inequalities, whenever  $p \in [1, 2]$  and  $q \in [2, \infty)$ .

The oscillation estimates immediately yield the norm equivalence statement (1.4). However, the norm equivalence itself may or may not be sharp; at this point we can only show sharpness for  $p > 2$ .

The principal step in getting oscillation estimates is to obtain sharp two-sided inequalities for the  $p$ -th power of  $L^p(Q)$ -norms of BMO functions, i.e. expressions of the form  $\langle |\varphi|^p \rangle_Q$ . It turns out that these quantities are always finite, meaning that  $\text{BMO}(Q) \subset L^p(Q)$ , as sets. These sharp  $L^p$  estimates on BMO are our main goal.

Following the template of the John–Nirenberg project [SV], we define the *upper and lower Bellman functions* for the problem: for  $p > 0$  and  $\varepsilon > 0$ , let

$$(1.5) \quad B_{\varepsilon,p}(x) = \sup_{\varphi \in \text{BMO}_\varepsilon(Q)} \left\{ \langle |\varphi|^p \rangle_Q : \langle \varphi \rangle_Q = x_1, \langle \varphi^2 \rangle_Q = x_2 \right\},$$

$$(1.6) \quad b_{\varepsilon,p}(x) = \inf_{\varphi \in \text{BMO}_\varepsilon(Q)} \left\{ \langle |\varphi|^p \rangle_Q : \langle \varphi \rangle_Q = x_1, \langle \varphi^2 \rangle_Q = x_2 \right\},$$

where  $\text{BMO}_\varepsilon(Q)$  is the  $\varepsilon$ -ball in  $\text{BMO}(Q)$ ,

$$\text{BMO}_\varepsilon(Q) = \{\varphi \in \text{BMO}(Q) : \|\varphi\|_{\text{BMO}(Q)} \leq \varepsilon\}.$$

It is easy to check that these functions are independent of the interval  $Q$ . Their domain is

$$\Omega_\varepsilon = \{x : x_1^2 \leq x_2 \leq x_1^2 + \varepsilon^2\}.$$

Indeed, for every  $\varphi \in \text{BMO}_\varepsilon(Q)$  and every subinterval  $J$  of  $Q$  the corresponding *Bellman point*  $(\langle \varphi \rangle_J, \langle \varphi^2 \rangle_J)$  is in  $\Omega_\varepsilon$ : the first inequality is just Hölder's inequality and the second one holds since  $\|\varphi\|_{\text{BMO}(Q)} \leq \varepsilon$ . This is the same domain as in [SV], and, as was the case there, each Bellman function will satisfy a Monge–Ampère equation on  $\Omega_\varepsilon$ . However, unlike the Bellman functions in [SV], the functions defined by (1.5) and (1.6) do not have additive homogeneity, while the domain  $\Omega_\varepsilon$  does not allow for multiplicative homogeneity. Thus we cannot hope to reduce the partial differential equations to ordinary ones, as was done in the previous work. Instead, we construct special foliations of  $\Omega_\varepsilon$  by straight-line characteristics along which each Bellman function must be linear. These foliations, different for various ranges of  $p$ , allow us to solve the PDE, thus obtaining the Bellman functions, and, simultaneously, construct optimizers in the inequalities being proved.

Having introduced the main objects of study, let us say a bit about the method. The term “Bellman functions” alludes to similar extremal constructs in the dynamic programming of R. Bellman [Be]. There are deep parallels between the Bellman frameworks of optimal stochastic control and harmonic analysis (see, for example, [NTV3] and [Vol]), although we make no use of those connections here. In the mid- to late 1980s, R. Burkholder ([B1, B2]) started using specially designed functions with delicate size and concavity properties to prove sharp inequalities for martingales by induction on scales. In the 1990s, the method took its modern form in the work of F. Nazarov, S. Treil, and A. Volberg, starting with [NTV1], [NT], and [NTV2]. However, the first exact Bellman functions, as explicit solutions of extremal problems such as (1.5) and (1.6), did not appear until [V1]. Bellman analysis on BMO originated with [SV] and continued in [V2]. Other notable explicit Bellman functions appeared in the work

of A. Melas and co-authors (see, for instance, [M1, M2, MN]), although those functions were found using combinatorial analysis of the operator in question, the dyadic maximal function, as opposed to solving the Bellman partial differential equation. The Bellman functions for the maximal operator were taken up again in [SSt] and [SSV]. Those works, together with [VV2], initiated the study of the relationship among explicit Bellman functions, the Monge–Ampère geometry of Bellman domains, and the structure of optimizers in the corresponding inequalities. Most recently, that line of investigation continued in [VV1]. The current work also presents that unified vision.

How do we proceed from estimates on  $\langle |\varphi|^p \rangle_Q$  to the norm equivalence statement (1.4)? By the definitions of  $\mathbf{B}_\varepsilon$  and  $\mathbf{b}_\varepsilon$ , we have, for every  $\varphi \in \text{BMO}_\varepsilon(Q)$  and every subinterval  $I$  of  $Q$ ,

$$\mathbf{b}_{\varepsilon,p}(\langle \varphi \rangle_I, \langle \varphi^2 \rangle_I) \leq \langle |\varphi|^p \rangle_I \leq \mathbf{B}_{\varepsilon,p}(\langle \varphi \rangle_I, \langle \varphi^2 \rangle_I).$$

Replacing  $\varphi$  with  $\varphi - \langle \varphi \rangle_I$  gives sharp oscillation estimates:

$$\mathbf{b}_{\varepsilon,p}(0, \langle \varphi^2 \rangle_I - \langle \varphi \rangle_I^2) \leq \langle |\varphi - \langle \varphi \rangle_I|^p \rangle_I \leq \mathbf{B}_{\varepsilon,p}(0, \langle \varphi^2 \rangle_I - \langle \varphi \rangle_I^2).$$

Analyzing each inequality separately, we get the desired norm inequalities:

$$\mathbf{b}_{\|\varphi\|,p}(0, \|\varphi\|) \leq \sup_{\text{interval } I \subset Q} \langle |\varphi - \langle \varphi \rangle_I|^p \rangle_I \leq \mathbf{B}_{\|\varphi\|,p}(0, \|\varphi\|),$$

where  $\|\cdot\| \stackrel{\text{def}}{=} \|\cdot\|_{\text{BMO}(Q)}$ .

The functions  $\mathbf{B}_{\varepsilon,p}$  and  $\mathbf{b}_{\varepsilon,p}$  can be viewed as special cases in a more general framework. Namely, take a function  $f$  on  $\mathbb{R}$  and define, formally, the Bellman functions

$$(1.7) \quad \mathbf{B}_{\varepsilon,f}(x) = \sup_{\varphi \in \text{BMO}_\varepsilon(Q)} \{ \langle f(\varphi) \rangle_I : \langle \varphi \rangle_I = x_1, \langle \varphi^2 \rangle_I = x_2 \}$$

and

$$(1.8) \quad \mathbf{b}_{\varepsilon,f}(x) = \inf_{\varphi \in \text{BMO}_\varepsilon(Q)} \{ \langle f(\varphi) \rangle_I : \langle \varphi \rangle_I = x_1, \langle \varphi^2 \rangle_I = x_2 \}.$$

(Here we explicitly allow for the possibility that one or both of these functions take on infinite values.) We will see that such a general view is beneficial: we will develop several canonical building blocks, each defined in a sub-domain of  $\Omega_\varepsilon$ ; these blocks, when appropriately arranged and glued, produce the functions (1.7), (1.8) for various choices of  $f$ , including the power function. Equally important, the optimizers in the Bellman functions — those functions  $\varphi$  on which supremum or infimum is attained — turn out to be determined locally by the canonical blocks, rather than by what specific  $f$  is being considered. In a very tangible sense, many BMO inequalities have the same optimizers.

Let us outline several important choices of  $f$ :  $f(s) = |s|^p$  yields definitions (1.5) and (1.6);  $f(s) = e^s$  produces the Bellman setup for the integral John–Nirenberg inequality from [SV]; since  $\lim_{p \rightarrow 0} \langle |\varphi|^p \rangle_Q^{1/p} = e^{\langle \log |\varphi| \rangle_Q}$ , the choice  $f(s) = \log |s|$  gives the appropriate limiting setup for (1.5) and (1.6), although we will see that the resulting “BMO<sup>0</sup>,” is not, in fact, BMO.

There are at least two more choices of importance:  $f(s) = \chi_{(-\infty, -\lambda] \cup [\lambda, \infty)}(s)$  for  $\lambda \geq 0$  and  $f(s) = e^{|s|}$ . The former yields the setup for the classical, weak-form John–Nirenberg inequality and the latter, for the two-sided integral John–Nirenberg inequality. Each of these two cases requires a slight modification of our building blocks. The first was considered in [V2], in a much more limited context, and the second will be considered elsewhere.

What are the assumed conditions on  $f$ ? Since our focus in the present work is on two-sided inequalities, we will assume that  $f$  is even. In order to simplify exposition we will also assume that  $f$  is non-negative and smooth, except, possibly, at 0.

Although all definitions in this section are valid in any dimension, at present we are only able to find the Bellman functions in the one-dimensional case, where all cubes in the definition

of BMO are intervals. We can obtain meaningful dimension-dependent estimates of the norm-equivalence constants in higher dimensions, but it is apart from our main interest here, which is in sharpness and explicit Bellman functions. In fact, we hold out the possibility that the Bellman functions — and so the norm estimates — are *dimension-free*, which would have major implications for analysis on BMO and related function classes (such as  $A_p$ ). At this time, we cannot show it, since a key geometric ingredient in our proofs works only in dimension 1; we hope to be able to give a definitive answer for higher dimensions in the future.

The geometry of Monge–Ampère foliations plays a central role in the construction of Bellman functions and their optimizers. As such, it is given a central role in our exposition. The picture of a foliation concisely captures the nature of the extremal problem at hand, and it is those foliations, first built locally and then carefully glued together, that we would like the reader to remember. While we provide precise algebraic descriptions of the Bellman functions, our proofs often appeal to their geometric nature. For example, when building optimizers, it is certainly possible to show that a given function is in  $\text{BMO}_\varepsilon$  by a direct calculation. However, it is more geometrically meaningful — and often simpler — to show that all of its Bellman points are in  $\Omega_\varepsilon$ .

Lastly, proving sharp norm estimates, while a lofty goal, does not require one to know the origin of the Bellman functions or of their optimizers; a relatively straightforward verification that these are, in fact, optimal would suffice. However, we choose to present their construction in considerable detail. This serves our second major goal: to provide a complete account of the modern Bellman–Monge–Ampère approach to problems with non-convex Bellman domains. Many other traditional harmonic analysis questions give rise to such domains and much of what follows should be applicable there.

The paper is organized as follows: in Section 2, we state the sharp inequalities proving which was our main motivation; in Section 3, we derive the boundary value problems our Bellman functions should solve and outline the geometric approach to finding the solutions; in Section 4 we construct local Bellman *candidates* for several subdomains of  $\Omega_\varepsilon$ ; these are glued together to produce global solutions in Section 5; in Sections 6 and 7 we use induction on scales and optimizers to prove that the global candidates are, in fact, the true Bellman functions; in Section 8 we provide the proofs of the inequalities from Section 2; finally, in Section 9, we briefly consider several additional choices of  $f$  in (1.7) and (1.8) and state the corresponding Bellman functions.

## 2. SHARP INEQUALITIES

Although the main results of this work are the explicit expressions for the functions (1.5) and (1.6) for various ranges of  $p$ , these are relatively complicated and stated in Section 5, after the relevant notation has been introduced and various blocks that comprise these expressions have been developed. In this section, we state the immediate consequences of our knowing the Bellman functions: the appropriate sharp inequalities for BMO. Since we know both the upper and lower functions, we naturally get two-sided inequalities; even though some are elementary, they are included for symmetry and to emphasize their source.

**Theorem 2.1.** *For an interval  $Q$  and any  $\varphi \in \text{BMO}(Q)$  such that  $\|\varphi\|_{\text{BMO}(Q)} \neq 0$  we have:*

$$\text{if } 0 < p \leq 1, \quad 2^{p-2} \|\varphi\|_{\text{BMO}(Q)}^{p-2} (\langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2) \leq \langle |\varphi - \langle \varphi \rangle_Q|^p \rangle_Q \leq \left( \langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2 \right)^{p/2};$$

$$\text{if } 1 \leq p \leq 2, \quad \frac{p}{2} \Gamma(p) \|\varphi\|_{\text{BMO}(Q)}^{p-2} (\langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2) \leq \langle |\varphi - \langle \varphi \rangle_Q|^p \rangle_Q \leq \left( \langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2 \right)^{p/2};$$

$$\text{if } 2 \leq p < \infty, \quad \left( \langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2 \right)^{p/2} \leq \langle |\varphi - \langle \varphi \rangle_Q|^p \rangle_Q \leq \frac{p}{2} \Gamma(p) \|\varphi\|_{\text{BMO}(Q)}^{p-2} (\langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2),$$

and these inequalities are sharp and attainable.

As a straightforward corollary of this theorem we obtain inequalities relating  $\text{BMO}^2$  and  $\text{BMO}^p$  norms; for  $p > 2$  these are the best possible.

**Theorem 2.2.** *For an interval  $Q$  and  $\varphi \in \text{BMO}(Q)$ , we have:*

$$\text{if } 0 < p \leq 1, \quad 2^{1-2/p} \|\varphi\|_{\text{BMO}(Q)} \leq \|\varphi\|_{\text{BMO}^p(Q)} \leq \|\varphi\|_{\text{BMO}(Q)};$$

$$\text{if } 1 \leq p \leq 2, \quad \left(\frac{p}{2} \Gamma(p)\right)^{1/p} \|\varphi\|_{\text{BMO}(Q)} \leq \|\varphi\|_{\text{BMO}^p(Q)} \leq \|\varphi\|_{\text{BMO}(Q)};$$

$$\text{if } 2 \leq p < \infty, \quad \|\varphi\|_{\text{BMO}(Q)} \leq \|\varphi\|_{\text{BMO}^p(Q)} \leq \left(\frac{p}{2} \Gamma(p)\right)^{1/p} \|\varphi\|_{\text{BMO}(Q)}.$$

The right-hand side inequalities for  $p < 2$  and both left- and right-hand side inequalities for  $p > 2$  are sharp and attainable.

*Remark 2.3.* At this point we do not know if the left-hand inequalities for  $p < 2$  are sharp.

Of course, Theorem 2.2 allows us to relate different  $\text{BMO}^p$  norms bypassing  $\text{BMO}^2$ , but the resulting inequalities may no longer be sharp. However, if we instead use Theorem 2.1 and our knowledge of the optimizers in its statements, we get the best possible inequalities relating  $p$ -oscillations for certain ranges of the parameter  $p$ . Specifically, we have

**Theorem 2.4.** *Fix an interval  $Q$  and numbers  $p_1 \in [1, 2]$  and  $p_2 \in [2, \infty)$ . Then, for any  $\varphi \in \text{BMO}(Q)$ , we have*

$$\langle |\varphi - \langle \varphi \rangle_Q|^{p_1} \rangle_Q^{p_2/p_1} \leq \langle |\varphi - \langle \varphi \rangle_Q|^{p_2} \rangle_Q \leq \frac{p_2 \Gamma(p_2)}{p_1 \Gamma(p_1)} \|\varphi\|_{\text{BMO}(Q)}^{p_2-p_1} \langle |\varphi - \langle \varphi \rangle_Q|^{p_1} \rangle_Q,$$

and these inequalities are sharp and attainable.

As mentioned in the introduction, in this paper we do not attempt to prove sharp two-sided John–Nirenberg inequality, that is we do not find the Bellman functions (1.7) and (1.8) for  $f(s) = e^{|s|}$ . However, because  $e^{|s|} = 1 + |s| + \sum_{k=2}^{\infty} \frac{|s|^k}{k!}$  and all upper Bellman functions  $\mathbf{B}_{p,\varepsilon}$  for  $p \geq 2$  turn out to have the same optimizers, we can sharply estimate

$$\langle e^{|\varphi - \langle \varphi \rangle_Q|} \rangle_Q - \langle |\varphi - \langle \varphi \rangle_Q| \rangle_Q.$$

Coupling the resulting estimate with the inequality  $\langle |\varphi - \langle \varphi \rangle_Q| \rangle_Q \leq \|\varphi\|_{\text{BMO}(Q)}$ , we obtain the following “almost sharp,” two-sided integral John–Nirenberg inequality:

**Theorem 2.5.** *Take an interval  $Q$  and let  $\varphi \in \text{BMO}(Q)$  be such that  $\|\varphi\| = \|\varphi\|_{\text{BMO}(Q)} < 1$ . Then we have*

$$\langle e^{|\varphi - \langle \varphi \rangle_Q|} \rangle_Q \leq C(\|\varphi\|).$$

Here the bound 1 on  $\|\varphi\|_{\text{BMO}}$  is sharp and the best (smallest) value of  $C(\|\varphi\|)$  satisfies

$$(2.1) \quad \frac{1 - \frac{\|\varphi\|}{2}}{1 - \|\varphi\|} \leq C(\|\varphi\|) \leq \frac{1 - \frac{\|\varphi\|^2}{2}}{1 - \|\varphi\|}.$$

By “sharp” we mean that there exist functions  $\varphi$  with norm 1 for which the inequality fails. The reader can compare this result with the one in [SV], where the sharp *one-sided* John–Nirenberg inequality was proved. One can get a sub-optimal two-sided inequality from a one-sided one by simply doubling the constant. If we do that with the result in [SV], we get

$$C(\|\varphi\|) \leq \frac{2e^{-\|\varphi\|}}{1 - \|\varphi\|},$$

which is worse than in (2.1).

It will be easy to prove all the inequalities stated in this section after the explicit Bellman functions (1.5) and (1.6), as well as their optimizers, have been found. We provide the proofs in Section 8.

### 3. BOUNDARY VALUE PROBLEMS AND MONGE–AMPÈRE FOLIATIONS

**3.1. The equations and boundary conditions.** Take an interval  $I$  and split it into two non-intersecting subintervals:  $I = I_- \cup I_+$ . For any sufficiently integrable function  $\varphi$  on  $I$ ,

$$\langle f(\varphi) \rangle_I = \alpha_- \langle f(\varphi) \rangle_{I_-} + \alpha_+ \langle f(\varphi) \rangle_{I_+},$$

where  $\alpha_{\pm} = |I_{\pm}|/|I|$ . Fix two points  $x^-, x^+ \in \Omega_{\varepsilon}$  such that  $\alpha_- x^- + \alpha_+ x^+ \in \Omega_{\varepsilon}$  and consider a sequence of functions  $\{\varphi_n\}$  such that  $\varphi_n \in \text{BMO}_{\varepsilon}(I_-) \cap \text{BMO}_{\varepsilon}(I_+)$ ,  $(\langle \varphi_n \rangle_{I_{\pm}}, \langle \varphi_n^2 \rangle_{I_{\pm}}) = x^{\pm}$ , and  $\lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_{I_{\pm}} = \mathbf{B}_{\varepsilon, f}(x^{\pm})$ . This gives

$$\lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_I = \alpha_- \mathbf{B}_{\varepsilon, f}(x^-) + \alpha_+ \mathbf{B}_{\varepsilon, f}(x^+).$$

If each  $\varphi_n$  could be chosen so that  $\varphi_n \in \text{BMO}_{\varepsilon}(I)$ , then we could conclude that

$$(3.1) \quad \mathbf{B}_{\varepsilon, f}(\alpha_- x^- + \alpha_+ x^+) \geq \alpha_- \mathbf{B}_{\varepsilon, f}(x^-) + \alpha_+ \mathbf{B}_{\varepsilon, f}(x^+).$$

Although  $\alpha_- x^- + \alpha_+ x^+ \in \Omega_{\varepsilon}$ , in general

$$\text{BMO}_{\varepsilon}(I) \subsetneq \text{BMO}_{\varepsilon}(I_-) \cap \text{BMO}_{\varepsilon}(I_+) \cap \{\varphi: \langle \varphi^2 \rangle_I - \langle \varphi \rangle_I^2 \leq \varepsilon^2\},$$

which is a major difference between continuous and dyadic BMO. We will, nonetheless, enforce condition (3.1) for the upper Bellman function candidate  $B$  and the converse inequality for the lower candidate  $b$ . Thus, we look for functions  $B$  and  $b$  with the property that for all pairs of points  $x^{\pm} \in \Omega_{\varepsilon}$  such that the whole line segment  $[x^-, x^+]$  is in  $\Omega_{\varepsilon}$ , we have

$$B(\alpha_- x^- + \alpha_+ x^+) \geq \alpha_- B(x^-) + \alpha_+ B(x^+)$$

and

$$b(\alpha_- x^- + \alpha_+ x^+) \leq \alpha_- b(x^-) + \alpha_+ b(x^+),$$

for all  $\alpha_{\pm} > 0$  with  $\alpha_+ + \alpha_- = 1$ . In other words, we look for  $B$  and  $b$  that are concave and, respectively, convex on any convex portion of  $\Omega_{\varepsilon}$ . We will refer to such functions as *locally concave* and *locally convex*, respectively. If we also assume sufficient differentiability on  $B$  and  $b$ , we get differential analogs of these finite-difference inequalities:

$$-\frac{d^2 B}{dx^2} \geq 0, \quad \frac{d^2 b}{dx^2} \geq 0 \quad \text{in } \Omega_{\varepsilon}.$$

In yet another restriction, the way Bellman function candidates are used in subsequent proofs suggests that we need to require that the candidates' concavity/convexity be degenerate, i.e. we require that

$$\det \left( \frac{d^2 B}{dx^2} \right) = 0, \quad \det \left( \frac{d^2 b}{dx^2} \right) = 0 \quad \text{in } \Omega_{\varepsilon}.$$

We have natural boundary conditions for the candidates. Observe that if  $\langle \varphi^2 \rangle_I = \langle \varphi \rangle_I^2$ , then  $\varphi$  is constant on  $I$  and so  $\langle f(\varphi) \rangle_I = f(x_1)$ , giving  $\mathbf{B}(x_1, x_1^2) = f(x_1)$  and  $\mathbf{b}(x_1, x_1^2) = f(x_1)$ . In addition, since  $f$  is assumed even, the Bellman functions (1.7) and (1.8) do not change if we replace  $\varphi$  with  $-\varphi$  in their definitions and so

$$\mathbf{B}(x_1, x_2) = \mathbf{B}(|x_1|, x_2), \quad \mathbf{b}(x_1, x_2) = \mathbf{b}(|x_1|, x_2).$$

Accordingly, it is enough to construct candidates  $B, b$  in the half-domain  $\Omega_{\varepsilon}^+ \stackrel{\text{def}}{=} \Omega_{\varepsilon} \cap \{x_1 \geq 0\}$ . Because of the symmetry, we impose a zero Neumann condition on the “internal” boundary  $x_1 = 0$ .

In what follows we will use the notation  $g_z \stackrel{\text{def}}{=} \frac{\partial g}{\partial z}$  for any function  $g$  and variable  $z$ . Thus, we set out to solve the following boundary value problems for candidates  $B$  and  $b$ :

$$(3.2) \quad \begin{aligned} B_{x_1 x_1} B_{x_2 x_2} &= B_{x_1 x_2}^2, & B_{x_1 x_1} &\leq 0, \quad B_{x_2 x_2} \leq 0 & \text{ in } \Omega_\varepsilon^+; \\ B(x_1, x_1^2) &= f(x_1), & B_{x_1}|_{x_1=0} &= 0, \end{aligned}$$

$$(3.3) \quad \begin{aligned} b_{x_1 x_1} b_{x_2 x_2} &= b_{x_1 x_2}^2, & b_{x_1 x_1} &\geq 0, \quad b_{x_2 x_2} \geq 0 & \text{ in } \Omega_\varepsilon^+; \\ b(x_1, x_1^2) &= f(x_1), & b_{x_1}|_{x_1=0} &= 0. \end{aligned}$$

**3.2. Monge–Ampère equations and their solutions.** In this part, we state a general result that will help us solve equations (3.2) and (3.3). A homogeneous Monge–Ampère equation says that at every point in the domain the Gaussian curvature of the surface defined by the solution is zero, i.e. there is a direction along which the solution is linear to the second order. Such directions form a vector field and the integral curves of this vector field turn out to be straight lines. These linear trajectories foliate the whole domain and — unless the defect of the Hessian is allowed to exceed 1 at a particular point — do not intersect.

There exists a formal way of obtaining Monge–Ampère solutions as functions that are linear along certain trajectories. From this viewpoint, describing a foliation of the domain by such trajectories (*a Bellman foliation*, in our parlance) is equivalent to determining the corresponding solution uniquely. The theorem below formalizes this method of foliations. We only consider the case of plane domains here. The general formulation, as well as a simple proof, can be found in [VV1].

**Theorem 3.1.** *Let  $\Omega$  be a plane domain and  $G = G(x_1, x_2)$  be a  $C^2$  function satisfying the homogeneous Monge–Ampère equation in  $\Omega$ :*

$$G_{x_1 x_1} G_{x_2 x_2} = G_{x_1 x_2}^2,$$

*and such that either  $G_{x_1 x_1} \neq 0$  or  $G_{x_2 x_2} \neq 0$ .*

*Let*

$$(3.4) \quad t_1 = G_{x_1}; \quad t_2 = G_{x_2}; \quad t_0 = G - t_1 x_1 - t_2 x_2.$$

*Then the functions  $t_k$  are constant on each integral trajectory generated by the kernel of the Hessian  $\frac{d^2 G}{dx^2}$ . Moreover, these integral trajectories are straight lines given by*

$$(3.5) \quad x_1 dt_1 + x_2 dt_2 + dt_0 = 0.$$

In what follows we will refer to the trajectories (3.5) as *extremal trajectories*, since they are used not only to find a suitable Bellman function candidate, but also to build extremizing functions/sequences proving that the candidate is, indeed, optimal.

**3.3. Main empirical principles.** According to the preceding discussion, we need to look for a candidate  $G = G(x_1, x_2)$  satisfying either (3.2) or (3.3). It would be desirable to have the Bellman function for each  $f$  given by a single  $C^2$  expression that is either concave or convex in the whole  $\Omega_\varepsilon$ . However, as we will see shortly, this is asking for too much. Instead, we will have our Bellman functions built out of several canonical Monge–Ampère solutions, each  $C^2$  in a portion of  $\Omega_\varepsilon$ , glued together so as to preserve the sign of the generalized second differential. A word about the nomenclature:  $G$  will stand for a generic Bellman candidate, whether upper or lower, whether defined on the whole domain or its part; as before,  $B$  will stand for a similarly generic upper candidate, and  $b$  for a lower one; when the emphasis is on the geometry of a canonical solution, the solution will be designated with its own name and necessary indices, as will be the sub-domain on which the solution is built; finally, we will omit the indices  $f$  and  $\varepsilon$  when no ambiguity arises.

We will build our canonical solutions by constructing the foliations of the corresponding portions of the domain by extremal trajectories, while adhering to several empirical principles. Since the Monge–Ampère apparatus (3.4)–(3.5) does not differentiate among various solutions of (3.2) or (3.3), additional arguments, having to do with the nature of the Bellman function, are needed, chief among which is how extremal trajectories are used to construct optimizers. These are our principles:

*Principle 1: Symmetry.* Since each Bellman function is even in  $x_1$ , each Bellman foliation is symmetric with respect to the line  $x_1 = 0$ .

Accordingly, we only need to build each Bellman foliation in the half-domain  $\Omega_\varepsilon^+$ , but have to consider carefully what happens on the internal boundary  $x_1 = 0$ . In which way can a trajectory intersect this boundary? Since the picture is symmetric, any such trajectory will have its counterpart from the domain  $\Omega_\varepsilon \cap \{x_1 \leq 0\}$  hitting the same point  $(0, x_2)$ . When the domain  $\Omega_\varepsilon$  is considered as a whole, we will have two trajectories intersecting at a point. This means that either they are two halves of the same horizontal trajectory or the Hessian has defect 2 at the point, meaning it is the identically zero matrix, and so the Bellman function is a linear function near that point. Indeed, we will encounter both situations below. In either case, since  $G_{x_1} = t_1 = 0$  when  $x_1 = 0$ , the function  $G$  depends only on  $x_2$  in the subdomain that includes the line  $x_1 = 0$ .

*Principle 2: Tangency.* Any trajectory intersecting the upper boundary of  $\Omega_\varepsilon$ ,  $x_2 = x_1^2 + \varepsilon^2$ , must do so tangentially (unless the Bellman function is linear in the part of the domain containing the point of tangency, as in this case any straight line is an extremal trajectory).

Let us explain this principle: observe that once the Bellman foliation for the problem is determined, every point of the domain can be put on one of the trajectories. That point, say  $x$ , prescribes the averages over the interval  $Q$  of each of the functions over which the extremum is taken in definitions (1.5) and (1.6). If such a function is optimal (or close to optimal), then when the interval is split into subintervals,  $Q = Q_- \cup Q_+$ , the pairs of averages  $x^\pm$  should remain on the same trajectory. In such a split,  $x$  will be located between  $x^-$  and  $x^+$ . We must, therefore, have a tangential intersection of the trajectory with the upper boundary, otherwise, if we take  $x^\pm$  close enough to  $x$ , one of the endpoints will exit  $\Omega_\varepsilon$ .

*Principle 3. Optimality.* The upper Bellman function is the smallest locally concave solution of the Monge–Ampère equation, while the lower Bellman function is the largest locally convex solution.

This principle may be intuitively clear, and it will be rigorously demonstrated in Section 6, where Bellman induction on scales is used to show that any locally concave function  $B$  satisfying  $B(x_1, x_1^2) = f(x_1)$  is a pointwise majorant of  $\mathbf{B}_{f,\varepsilon}$ , while any locally convex solution  $b$  satisfying  $b(x_1, x_1^2) = f(x_1)$  is a pointwise minorant of  $\mathbf{b}_{f,\varepsilon}$ . The terms “super-solution” and “sub-solution,” respectively, are typically used for such candidates.

#### 4. LOCAL BELLMAN CANDIDATES: THE FOUR BUILDING BLOCKS

Having laid down our basic principles, we start building the foliations (and so the candidates) that comply with these principles. Since we know the behavior of any extremal trajectory touching the upper boundary, it is convenient to start with such trajectories. The tangent line at a point  $(a, a^2 + \varepsilon^2)$  is given by

$$(4.1) \quad x_2 = 2ax_1 + \varepsilon^2 - a^2.$$

Each such tangent intersects the lower boundary  $x_2 = x_1^2$  at two points,

$$u_\pm = a \pm \varepsilon.$$

Let us show that if the foliation being built includes a *family* of such tangents, the whole tangent line cannot be a single extremal trajectory, since  $G_{x_2x_2}$  changes sign depending on whether  $x$  is to the left or to the right of  $(a, a^2 + \varepsilon^2)$ . Indeed, for every  $x$  on the tangent



line (4.1),  $t_1$ ,  $t_2$ , and  $t_0$  are functions of  $a$  only. Recall that  $t_2 = G_{x_2}$ . Then  $G_{x_2 x_2} = t'_2(a)a_{x_2}$ . Since  $a$  is fixed along this line,  $t'_2(a)$  is constant. From (4.1),

$$(4.2) \quad a_{x_2} = \frac{1}{2(x_1 - a)},$$

which changes sign at  $x_1 = a$ .

Therefore, if the foliation contains a family of such tangents (which justifies differentiating with respect to  $a$  above), each extremal trajectory continues either to the right of the point of tangency or to the left, but not both. This is in contrast to the situation, also considered below, when the sub-domain being foliated lies entirely under *one* two-sided tangent.

We will now consider the four sub-foliations out of which we will later build two complete Bellman foliations (one for the upper and one for the lower function) for various ranges of  $p$ . The sub-foliations are two families of one-sided tangents and two “phase transition regimes” used to connect those families smoothly. The four are, in order of presentation:

- (1) A collection of one-sided tangents for which the point of their intersection with the lower boundary is to the right of the point of tangency;
- (2) A collection of one-sided tangents whose point of intersection with the lower boundary is to the left of the point of tangency;
- (3) Any foliation of the convex compact set lying under a single two-sided tangent. There are several ways to foliate such a set, depending on its location; in each case the bounding tangent is an element of the foliation;
- (4) Any foliation of a curvilinear “triangle” located between two differently oriented one-sided tangents sharing a point on the lower boundary. As we will see, in such a triangle the Bellman function must be linear and so the very notion of a Bellman foliation is trivial, as every straight line is an extremal trajectory.

*Remark 4.1.* It would, of course, be preferable to not have to combine various solutions, instead having a single family of tangents as the Bellman foliation on each side of the line  $x_1 = 0$  with a single transition regime containing that line and connecting the two families. Such a situation does, in fact, occur for  $f(s) = |s|^p$ ,  $p \geq 1$ . However, for  $0 < p < 1$  the solutions corresponding to each family of tangents change concavity at various points throughout the domain, necessitating the introduction of other transition regimes.

**4.1. The tangents with  $u = a + \varepsilon$ .** Here we consider a family of tangents (4.1) with  $a \in [a_1, a_2]$ , or  $u \in [u_1, u_2]$ ,  $u_i = a_i + \varepsilon$ . The tangents under consideration are one-sided, extending to the right of the point of tangency, i.e. we have  $x_1 \in [a, u]$ . The Bellman candidate built along these trajectories will be called  $F^+$  (or, more fully,  $F^+(x; u_1, u_2)$ ) and the part of  $\Omega_\varepsilon$  so foliated,  $\Omega_{F^+}(u_1, u_2)$ . The foliation is shown in Figure 1.

The candidate  $F^+$  is linear along each trajectory (4.1) and satisfies  $F^+(u, u^2) = f(u)$ , thus,

$$(4.3) \quad F^+(x_1, 2ax_1 + \varepsilon^2 - a^2) = m(u)(x_1 - u) + f(u).$$

Let us calculate  $t_2 = F^+_{x_2}$ . Using (4.2) and the equality  $u_{x_2} = a_{x_2}$ , we get

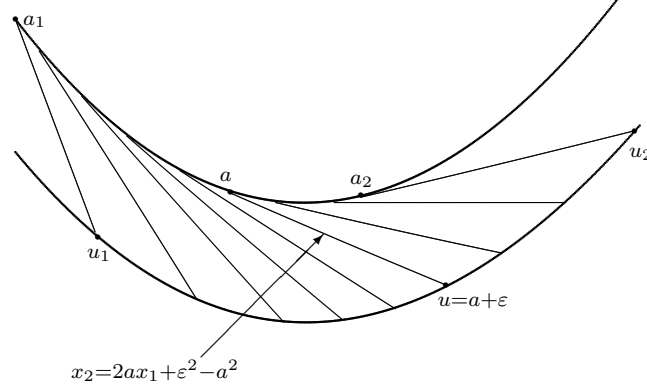
$$t_2 = \frac{m'(u)(x_1 - u) - m(u) + f'(u)}{2(x_1 - a)} = \frac{1}{2}m'(u) + \frac{f'(u) - \varepsilon m'(u) - m(u)}{2(x_1 - u + \varepsilon)}.$$

Since  $t_2$  is fixed whenever  $a$  (or  $u$ ) is fixed, this gives

$$(4.4) \quad t_2(u) = \frac{1}{2}m'(u)$$

and

$$(4.5) \quad \varepsilon m'(u) + m(u) = f'(u).$$

FIGURE 1. The foliation of  $\Omega_{F+}(u_1, u_2)$ .

Solving equation (4.5) yields

$$m(u) = e^{-u/\varepsilon} \left( C + \frac{1}{\varepsilon} \int_{u_1}^u f'(s) e^{s/\varepsilon} ds \right),$$

and so we obtain our solution, defined for  $x \in \Omega_{F+}(u_1, u_2)$ ,

$$(4.6) \quad F^+(x; u_1, u_2) = e^{-u/\varepsilon} \left( C + \frac{1}{\varepsilon} \int_{u_1}^u f'(s) e^{s/\varepsilon} ds \right) (x_1 - u) + f(u),$$

where  $u$  is given as a function of  $x$  by

$$(4.7) \quad u = u_+ = x_1 + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

The last equation comes from plugging  $u = a + \varepsilon$  into (4.1), and the minus sign in the front of the square root was chosen, because  $x_1 \in [a, u]$ .

We will be concerned with whether (4.6) gives a concave or convex candidate for each specific choice of  $f$  (and in what parts of the domain). To that end, let us check the sign of  $F_{x_2 x_2}^+$ . Using (4.4) and (4.5), we obtain

$$F_{x_2 x_2}^+ = t_2'(u) u_{x_2} = \frac{1}{2} m''(u) u_{x_2} = \frac{1}{2\varepsilon} (f''(u) - f'(u)/\varepsilon + m(u)/\varepsilon) u_{x_2}.$$

Since  $u_{x_2} > 0$  by (4.2), we conclude that  $\text{sgn}(F_{x_2 x_2}^+) = \text{sgn}(\tau_+)$ , where

$$(4.8) \quad \tau_+(u) \stackrel{\text{def}}{=} \varepsilon e^{u/\varepsilon} m''(u) = (f''(u) - f'(u)/\varepsilon) e^{u/\varepsilon} + \frac{1}{\varepsilon} C + \frac{1}{\varepsilon^2} \int_{u_1}^u f'(s) e^{s/\varepsilon} ds.$$

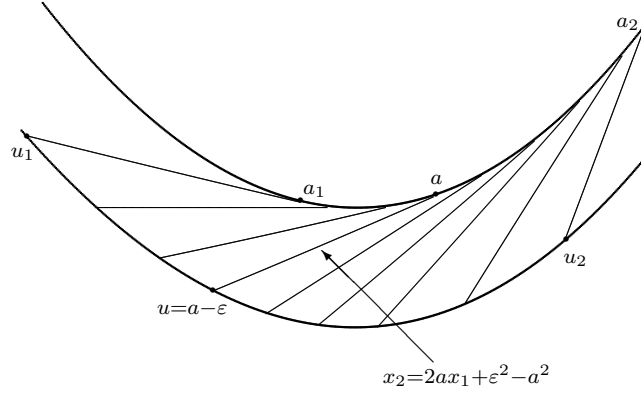
Integrating by parts twice gives

$$(4.9) \quad \tau_+(u) = (f''(u_1) - f'(u_1)/\varepsilon) e^{u_1/\varepsilon} + \frac{1}{\varepsilon} C + \int_{u_1}^u f'''(s) e^{s/\varepsilon} ds.$$

**4.2. The tangents with  $u = a - \varepsilon$ .** We now switch to considering those one-sided tangents whose point of intersection with the bottom boundary curve lies to the left of the point of tangency. The picture is symmetric with the previous one with respect to the  $x_2$ -axis. Thus, we replace in the preceding formulas  $x_1$  by  $-x_1$ ,  $u$  by  $-u$ ,  $a$  by  $-a$ ,  $f(s)$  by  $f(-s)$ ,  $u_1$  by  $-u_2$ , and  $u_2$  by  $-u_1$ . It is possible, of course, to derive all formulas independently, but we only indicate which changes are necessary. The solution constructed will be called  $F^-$  (alternatively,  $F^-(x; u_1, u_2)$ ) and the portion of  $\Omega_\varepsilon$  being foliated,  $\Omega_{F-}(u_1, u_2)$ . The foliation is shown on Figure 2.

Equations (4.3) and (4.4) remain the same, but equation (4.5) changes to

$$(4.10) \quad \varepsilon m'(u) - m(u) = -f'(u),$$

FIGURE 2. The foliation of  $\Omega_{F^-}(u_1, u_2)$ .

which gives

$$m(u) = e^{u/\varepsilon} \left( C + \frac{1}{\varepsilon} \int_u^{u_2} f'(s) e^{-s/\varepsilon} ds \right).$$

This yields our canonical solution, defined for  $x \in \Omega_{F^-}(u_1, u_2)$  :

$$F^-(x; u_1, u_2) = e^{u/\varepsilon} \left( C + \frac{1}{\varepsilon} \int_u^{u_2} f'(s) e^{-s/\varepsilon} ds \right) (x_1 - u) + f(u),$$

with

$$u = u_- = x_1 - \varepsilon + \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

As before, the sign of  $F_{x_2 x_2}^-$  will be of interest. We have

$$F_{x_2 x_2}^- = t_2'(u) u_{x_2} = \frac{1}{2} m''(u) u_{x_2} = \frac{1}{2\varepsilon} (-f''(u) - f'(u)/\varepsilon + m(u)/\varepsilon) u_{x_2}.$$

Since  $u_{x_2} < 0$  by (4.2), we have  $\text{sgn}(F_{x_2 x_2}^-) = \text{sgn}(\tau_-)$ , where

$$(4.11) \quad \tau_-(u) \stackrel{\text{def}}{=} -\varepsilon e^{-u/\varepsilon} m''(u) = (f''(u) + f'(u)/\varepsilon) e^{-u/\varepsilon} - \frac{1}{\varepsilon} C - \frac{1}{\varepsilon^2} \int_u^{u_2} f'(s) e^{-s/\varepsilon} ds.$$

Integrating by parts twice gives

$$(4.12) \quad \tau_-(u) = (f''(u_2) + f'(u_2)/\varepsilon) e^{-u_2/\varepsilon} - \frac{1}{\varepsilon} C - \int_u^{u_2} f'''(s) e^{-s/\varepsilon} ds.$$

**4.3. The region under a two-sided tangent.** Next, fix  $a$  and consider the tangent  $x_2 = 2ax_1 + \varepsilon^2 - a^2$ . It intersects the lower boundary at the points  $(a \pm \varepsilon, (a \pm \varepsilon)^2)$  and bounds the convex subset of  $\Omega_\varepsilon$

$$\Omega_L(a) = \{x_1^2 \leq x_2 \leq 2ax_1 + \varepsilon^2 - a^2\}.$$

(In our nomenclature, each domain is designated by the name of the corresponding Bellman candidate; in a bit of inelegance, here we have to define and name the domain first. Naturally, we reserve the name  $L$  for each candidate built on  $\Omega_L$ .)

In order for any foliation of  $\Omega_L(a)$  to be a part of a symmetric foliation of the whole  $\Omega_\varepsilon$ , we must have either  $a = 0$  (and so the condition  $L_{x_1}|_{x_1=0} = 0$  must come into play) or  $|a| - \varepsilon \geq 0$ , i.e.  $\Omega_L(|a|) \subset \Omega_\varepsilon^+$ .

Let us first construct the solution for the set  $\Omega_L(0) = \{x_1^2 \leq x_2 \leq \varepsilon^2\}$ . The Bellman candidate here will be called  $L_0$ . As explained in the formulation of the symmetry principle,  $L_0$  must be a function of  $x_2$  only. Thus we have  $L_0(x) = g(x_2)$  for some function  $g$ . On the boundary,  $L_0(x_1, x_1^2) = g(x_1^2) = f(x_1)$  and so  $g(x_2) = f(\sqrt{x_2})$ . Thus,

$$(4.13) \quad L_0(x) = f(\sqrt{x_2}).$$

The corresponding foliation of  $\Omega_L(0)$  consists of horizontal lines.

We now fix an  $a \geq \varepsilon$  and construct a solution (there are actually two) in the region  $\Omega_L(a)$ . Assume that there is a family of trajectories foliating (a part of) this domain. Say such a trajectory intersects the lower boundary at the point  $(u, u^2)$  and we have a whole interval of such values  $u$ . We do not want our trajectories to intersect the upper boundary of  $\Omega_L(a)$ , the line  $x_2 = 2ax_1 + \varepsilon^2 - a^2$ , since we want to have that line as one of the trajectories. Thus each trajectory exits  $\Omega_L(a)$  through a point  $(v, v^2)$  on the lower boundary with  $v = v(u)$ . Then the trajectory is given by

$$(4.14) \quad x_2 = (v + u)x_1 - vu,$$

and we have, for a candidate  $L$ ,

$$L(x_1, (v + u)x_1 - vu) = m(u)(x_1 - u) + f(u).$$

On the other hand, we must have  $L(v, v^2) = f(v)$ , hence,

$$(4.15) \quad m = \frac{f(v) - f(u)}{v - u}.$$

As before, let us find  $t_2 = L_{x_2}$ . We have  $t_2 = (m'(x_1 - u) - m + f'(u))u_{x_2}$ . Using (4.14) gives

$$u_{x_2} = \frac{1}{(v' + 1)(x_1 - u) + u - v},$$

and so

$$t_2 = \frac{m'}{v' + 1} + \frac{m'(v - u)/(v' + 1) - m + f'(u)}{(v' + 1)(x_1 - u) + u - v},$$

which means

$$(4.16) \quad t_2 = \frac{m'}{v' + 1}, \quad m'(v - u) = (m - f'(u))(v' + 1).$$

The last equation, together with (4.15) and a bit of algebra, gives

$$\frac{f(v) - f(u)}{v - u} v' = \frac{f'(v) + f'(u)}{2} v'.$$

If  $v' \neq 0$ , then

$$\frac{f(v) - f(u)}{v - u} = \frac{f'(v) + f'(u)}{2},$$

which, for  $f(s) = |s|^p$ , is possible only if  $v = -u$ , i.e. only in the just-considered case of  $L_0$ .

Therefore,  $v'(u) = 0$  and so the trajectories enter through different points for different values of  $u$ , but exit through the same point  $(v, v^2)$ . Again, since we want the line  $x_2 = 2ax_1 + \varepsilon^2 - a^2$  as a trajectory, we should have the exit point in one of the corners of  $\Omega_L(a)$ , i.e. either  $v = a - \varepsilon = u_-$  or  $v = a + \varepsilon = u_+$ . In fact, each of these choices yields a solution that is either concave or convex in the whole region  $\Omega_L(a)$ .

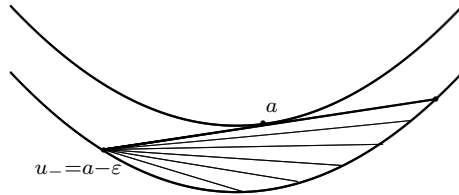
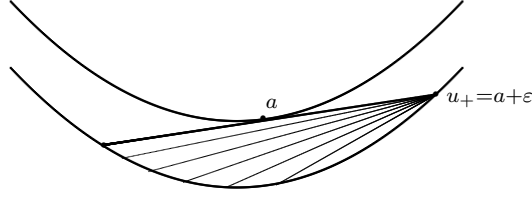


FIGURE 3. The region  $\Omega_L(a)$  foliated according to  $L^-$

FIGURE 4. The region  $\Omega_L(a)$  foliated according to  $L^+$ 

To determine whether the solution  $L$  is concave or convex for each choice of  $v$ , we compute, using (4.16) with  $v' = 0$ ,

$$L_{x_2 x_2} = m''(u)u_{x_2} = \frac{m''(u)}{x_1 - v}.$$

Thus  $\text{sgn}(L_{x_2 x_2}) = \text{sgn}[m''(u)/(x_1 - v)]$ . A simple calculation shows that  $m''(u) = \frac{1}{3}f'''(\xi)$  for some  $\xi$  between  $u$  and  $v$ . For  $v = u_-$  we have  $x_1 \geq v$ , while for  $v = u_+$  we have  $x_1 \leq v$ . Thus we obtain two solutions in  $\Omega_L(a)$ :

$$(4.17) \quad L^\pm(x; a) = \frac{f(u_\pm) - f(u)}{u_\pm - u}(x_1 - u_\pm) + f(u_\pm),$$

with  $u_\pm$  and  $u$  given by

$$(4.18) \quad u_\pm = a \pm \varepsilon, \quad u = \frac{x_2 - u_\pm x_1}{x_1 - u_\pm},$$

and

$$(4.19) \quad \text{sgn}(L_{x_2 x_2}^-) = \text{sgn}(f'''), \quad \text{sgn}(L_{x_2 x_2}^+) = -\text{sgn}(f''').$$

The last two identities make sense if  $f'''$  does not change its sign on the interval  $(a - \varepsilon, a + \varepsilon)$ , as is the case for  $f(s) = |s|^p$ .

**4.4. The region between two tangents.** Here we fix  $u$  and consider the region  $\Omega_T(u)$  between the two differently directed one-sided tangents sharing a common lower-boundary point  $(u, u^2)$  (see Figure 5):

$$\Omega_T(u) = \{u - \varepsilon \leq x_1 \leq u + \varepsilon, 2ux_1 - u^2 + 2\varepsilon|u - x_1| \leq x_2 \leq x_1^2 + \varepsilon^2\}.$$

As before, symmetry considerations dictate that for each  $\Omega_T(u)$  we must either have  $u = 0$  or  $\Omega_T(|u|) \subset \Omega_\varepsilon^+$  (that is  $|u| \geq \varepsilon$ ). In either case, the need for a solution — let us call it  $T = T(x; u)$  — in this region arises when we have to glue two foliations (typically, those for  $F^-$  and  $F^+$  or  $L^-$  and  $L^+$ ). This allows us to use the optimality principle: since each of the two tangents bounding  $\Omega_T(u)$  is an element of a Bellman foliation in a portion of  $\Omega_\varepsilon$ , the compound Bellman candidate being built is linear along each tangent. Since we are to construct either the smallest concave or the largest convex Monge–Ampère solution in  $\Omega_T(u)$ ,  $T$  is a linear function of  $x$ .

Let us first briefly consider the case  $u = 0$ . We are looking for a function

$$T_0(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0$$

on  $\Omega_T(0)$ . As discussed earlier,  $T_0$  is a function of  $x_2$  only and so  $\alpha_1 = 0$ . In addition  $T_0(0, 0) = f(0)$ , which gives

$$(4.20) \quad T_0(x) = \alpha x_2 + f(0).$$

The constant  $\alpha$  is determined by reading the boundary value off the tangent  $x_2 = 2\varepsilon x_1$ ; that value, in turn, depends on the other components of the global Bellman foliation. We will see in the next section how this simple step is accomplished.

The situation when  $u \neq 0$  is more involved. In theory, different choices of  $f$  may imply the need to use  $\Omega_T$  to glue various combinations of foliations of  $\Omega_{F^\pm}$  and  $\Omega_{L^\pm}$ , described in the earlier sections. Thus, in general we are looking for a linear candidate  $T$  in the form

$$(4.21) \quad T = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0,$$

such that it is equal to a particular candidate,  $G^-$ , along its left bounding tangent, the line  $x_2 = 2(u - \varepsilon)x_1 - u^2 + 2\varepsilon u$ , and to another candidate,  $G^+$ , along its right bounding tangent,  $x_2 = 2(u + \varepsilon)x_1 - u^2 - 2\varepsilon u$ . That is we want to ensure that

$$(4.22) \quad T(x^\pm; u) = G^\pm(x^\pm),$$

where  $x^-$  is any point on the left tangent and  $x^+$  is any point on the right one.

The left and right tangents are assumed to be extremal trajectories for  $G^-$  and  $G^+$ , respectively, and so each function is linear along the appropriate tangent. We already have equality at the corner  $(u, u^2)$ , so it is sufficient to glue our solutions at the other two corners of  $\Omega_T(u)$  :

$$(4.23) \quad \begin{aligned} G^-(u - \varepsilon, u^2 - 2\varepsilon u + 2\varepsilon^2) &= \alpha_1(u - \varepsilon) + \alpha_2(u^2 - 2\varepsilon u + 2\varepsilon^2) + \alpha_0, \\ G^+(u + \varepsilon, u^2 + 2\varepsilon u + 2\varepsilon^2) &= \alpha_1(u + \varepsilon) + \alpha_2(u^2 + 2\varepsilon u + 2\varepsilon^2) + \alpha_0. \end{aligned}$$

In this paper, the only situation where we encounter  $\Omega_T(u)$  with  $u \neq 0$ , is when it is used as a transition regime between  $\Omega_{F^+}(u_1, u)$  and  $\Omega_{F^-}(u, u_2)$  for some numbers  $u_1, u_2$ . It turns out that such a transition places an important restriction on  $u$ . Let us elaborate.

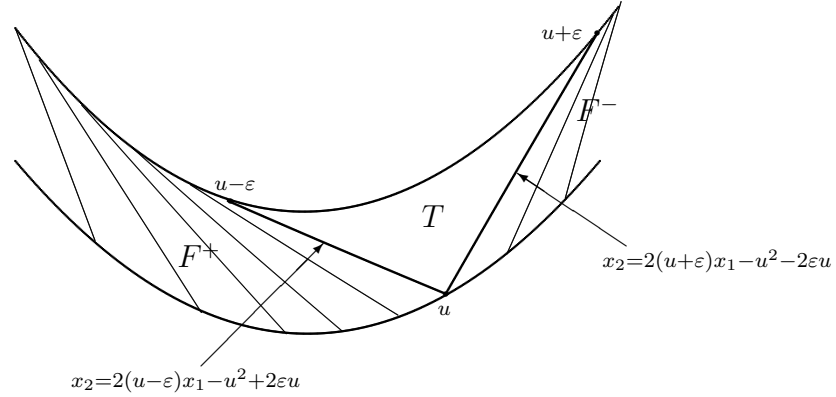


FIGURE 5. The region  $\Omega_T(u)$  connecting the foliations for  $F^+$  and  $F^-$ .

We are looking for a solution in the form (4.21) according to two requirements. The first one is that relations (4.23) be fulfilled, with  $G^- = F^-$  and  $G^+ = F^+$ . The second requirement is that the solution must preserve the sign of the (generalized) second derivative along any direction. This second requirement is, in fact, the reason why  $T$  is necessary: both  $F^+$  and  $F^-$  are continuous in the whole domain, but their various second derivatives change signs.

Recall that we can write  $F^+$  and  $F^-$  as

$$F^+ = m_+(u)(x_1 - u) + f(u), \quad F^- = m_-(u)(x_1 - u) + f(u),$$

where the coefficients  $m_+$  and  $m_-$  satisfy the differential equations (4.5) and (4.10), respectively:

$$(4.24) \quad m'_+ = -\frac{1}{\varepsilon}(m_+ - f'), \quad m'_- = \frac{1}{\varepsilon}(m_- - f').$$

For  $G^- = F^+$  and  $G^+ = F^-$ , (4.23) gives

$$\begin{aligned} m_+(u)(-\varepsilon) + f(u) &= \alpha_1(u - \varepsilon) + \alpha_2(u^2 - 2\varepsilon u + 2\varepsilon^2) + \alpha_0, \\ m_-(u)\varepsilon + f(u) &= \alpha_1(u + \varepsilon) + \alpha_2(u^2 + 2\varepsilon u + 2\varepsilon^2) + \alpha_0. \end{aligned}$$

Together with the boundary condition

$$f(u) = \alpha_1 u + \alpha_2 u^2 + \alpha_0$$

this yields

$$(4.25) \quad \begin{aligned} \alpha_1 &= \frac{m_+ + m_-}{2} - \frac{m_- - m_+}{2\varepsilon} u, \\ \alpha_2 &= \frac{m_- - m_+}{4\varepsilon}, \\ \alpha_0 &= -\frac{m_+ + m_-}{2} u + \frac{m_- - m_+}{4\varepsilon} u^2 + f. \end{aligned}$$

This determines the function  $T$  up to the parameter  $u$ , which is *not* free. It has to be chosen so that the Bellman candidate  $G$  constructed of the blocks  $F^+$ ,  $T$ , and  $F^-$  is either concave or convex in the combined domain  $\Omega_{F^+}(u_1, u) \cup \Omega_T(u) \cup \Omega_{F^-}(u, u_2)$ . We cannot enforce differentiability along the tangents bounding  $\Omega_T(u)$ , but can, and will, require that the jumps in the first derivative(s) be of the same sign on both boundaries. We will see in Section 6 that it is enough to check this in a single direction transversal to the boundary. Let us do so for  $G_{x_2}$ . Along the left and right bounding tangents the jump in this derivative is, respectively,  $\alpha_2 - F_{x_2}^+$  and  $\alpha_2 - F_{x_2}^-$ . Recall that each  $F_{x_2}^+$  and  $F_{x_2}^-$  has a constant value along the corresponding line, given, respectively, by

$$F_{x_2}^+ = -\frac{1}{2\varepsilon}(m_+ - f') \quad \text{and} \quad F_{x_2}^- = \frac{1}{2\varepsilon}(m_- - f'),$$

where we have used (4.4) and (4.24). Taking into account formula (4.25) for  $\alpha_2$ , we can write the compatibility condition as

$$\left[ \frac{m_- - m_+}{4\varepsilon} + \frac{1}{2\varepsilon}(m_+ - f') \right] \left[ \frac{m_- - m_+}{4\varepsilon} - \frac{1}{2\varepsilon}(m_- - f') \right] = -\frac{1}{16\varepsilon^2} [m_+ + m_- - 2f']^2 \geq 0,$$

and, finally, as

$$(4.26) \quad m_+(u) + m_-(u) = 2f'(u).$$

We see that, indeed, in gluing  $F^+$  and  $F^-$  via  $T$ , the parameter  $u$  cannot be chosen arbitrarily. In fact, only in certain circumstances such a value  $u$  exists. If it does, we can write the function  $T$  using (4.21), (4.25), and (4.26) as

$$(4.27) \quad T(x; u) = f'(u)(x_1 - u) + \frac{m_-(u) - f'(u)}{2\varepsilon} (-2x_1 u + x_2 + u^2) + f(u).$$

This completes the description of the four basic blocks out of which we will assemble global Bellman candidates. The next section is devoted to this task.

## 5. GLOBAL BELLMAN CANDIDATES

In this section, we construct a set of global Bellman candidates, i.e. candidates that have the same sign of the generalized second differential in the whole domain  $\Omega_\varepsilon$ . The main emphasis is on our specific choice of the embedding function  $f$ ,  $f(s) = |s|^p$ , although some results are stated in more generality, which will be useful a bit later.

Although the global Bellman foliations do significantly depend on the range of  $p$  considered, some useful statements can be made for all  $p$ . In the previous section, we built general candidates  $F^+(x; u_1, u_2)$  and  $F^-(x; u_1, u_2)$ , each determined up to a constant  $C$ . To specify this constant for  $F^+(x; u_1, u_2)$ , we need to know its left neighbor; for  $F^-(x; u_1, u_2)$ , we need

to know its right neighbor. If  $F^+$  has no left neighbor, i.e. if  $u_1 = -\infty$ , the constant is determined using limiting considerations, and similarly for  $F^-$  in the case  $u_2 = \infty$ .

From our limited arsenal of canonical blocks, only blocks of the  $L$  type can be located directly to the left of  $F^+(x; u_1, u_2)$ . This gives us a boundary condition for  $F^+$  on the line  $x_2 = 2(u_1 - \varepsilon)x_1 - u_1^2 + 2u_1\varepsilon$ , shared by the two canonical sub-domains. We know from (4.17) and (4.18) that

$$L(a, a^2 + \varepsilon^2; a) = \frac{f(a - \varepsilon) + f(a + \varepsilon)}{2}.$$

Setting  $a = u_1 - \varepsilon$  and equating the result with  $F^+(x; u_1, u_2)$  from (4.6) for  $u = u_1$  and  $x_1 = u_1 - \varepsilon$ , we obtain

$$C = \frac{f(u_1) - f(u_1 - 2\varepsilon)}{2\varepsilon} e^{u_1/\varepsilon},$$

which gives

$$(5.1) \quad F^+(x; u_1, u_2) = \frac{1}{\varepsilon} e^{-u/\varepsilon} \left[ \frac{f(u_1) - f(u_1 - 2\varepsilon)}{2} e^{u_1/\varepsilon} + \int_{u_1}^u f'(s) e^{s/\varepsilon} ds \right] (x_1 - u) + f(u),$$

where  $u$  is given as a function of  $x$  by (4.7).

Let us specify this formula for  $f(s) = |s|^p$  in two cases we will need:  $u_1 = \varepsilon$  and  $u_1 = 2\varepsilon$ . In the first case we have, after a change of variable in the integral,

$$(5.2) \quad F^+(x; \varepsilon, u_2) = p\varepsilon^{p-1} e^{-u/\varepsilon} \left[ \int_1^{u/\varepsilon} t^{p-1} e^t dt \right] (x_1 - u) + u^p, \quad x \in \Omega_{F^+}(\varepsilon, u_2),$$

and in the second one,

$$(5.3) \quad F^+(x; 2\varepsilon, u_2) = \varepsilon^{p-1} e^{-u/\varepsilon} \left[ 2^{p-1} e^2 + p \int_2^{u/\varepsilon} t^{p-1} e^t dt \right] (x_1 - u) + u^p, \quad x \in \Omega_{F^+}(2\varepsilon, u_2),$$

where

$$u = x_1 + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

We then have  $\text{sgn}(F_{x_2 x_2}^+) = \text{sgn}(\tau_+)$ , where, according to (4.9),

$$(5.4) \quad \tau_+(u) = p(p-2)\varepsilon^{p-2} \left[ e + (p-1) \int_1^{u/\varepsilon} t^{p-3} e^t dt \right],$$

if  $u_1 = \varepsilon$ , and

$$(5.5) \quad \tau_+(u) = (p-1)(p-2)\varepsilon^{p-2} \left[ 2^{p-2} e^2 + p \int_2^{u/\varepsilon} t^{p-3} e^t dt \right],$$

if  $u_1 = 2\varepsilon$ .

Now, let us consider the case when the block  $\Omega_{F^+}$  has no left neighbor, i.e. when  $u_1 = -\infty$ . The argument here is subtler. Namely, we first look at the sign of  $F_{x_2 x_2}^+$  and then invoke the optimality principle. For large negative  $u$  we have  $\text{sgn}(F_{x_2 x_2}^+) = \text{sgn}(\tau_+) = \text{sgn}(C)$ , with  $\tau_+$  given by (4.9). Thus, if  $C < 0$ , we have an upper Bellman candidate. Since we want the *smallest* upper candidate, and  $F^+$  given by (4.6) decreases as  $C$  grows, we set  $C = 0$ . On the other hand, if  $C > 0$ , we have a lower candidate, which we want to maximize; this, again, leads us to take  $C = 0$ . By symmetry,  $\Omega_{F^+}(-\infty, u_2)$  must be contained in  $\Omega_\varepsilon \cap \{x_1 \leq 0\}$  and so we can restrict ourselves to  $u_2 \leq 0$ . Therefore, we obtain the following expression:

$$(5.6) \quad F^+(x; -\infty, u_2) = p\varepsilon^{p-1} e^{-u/\varepsilon} \left[ \int_{-u/\varepsilon}^\infty t^{p-1} e^{-t} dt \right] (u - x_1) + |u|^p, \quad x \in \Omega_{F^+}(-\infty, u_2), \quad u \leq u_2 \leq 0,$$



with  $u$  given by

$$u = x_1 + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

The consideration for  $F^-(x; u_1, u_2)$  is entirely symmetrical: (5.1), (5.2), (5.3), and (5.6) become, respectively,

$$(5.7) \quad F^-(x; u_1, u_2) = \frac{1}{\varepsilon} e^{u/\varepsilon} \left[ \frac{f(u_2 + 2\varepsilon) - f(u_2)}{2} e^{-u_2/\varepsilon} + \int_u^{u_2} f'(s) e^{-s/\varepsilon} ds \right] (x_1 - u) + f(u),$$

$$F^-(x; u_1, -\varepsilon) = p\varepsilon^{p-1} e^{u/\varepsilon} \left[ \int_1^{-u/\varepsilon} t^{p-1} e^t dt \right] (u - x_1) + |u|^p, \quad x \in \Omega_{F^-}(u_1, -\varepsilon),$$

$$F^-(x; u_1, -2\varepsilon) = \varepsilon^{p-1} e^{-u/\varepsilon} \left[ 2^{p-1} e^2 + p \int_2^{-u/\varepsilon} t^{p-1} e^t dt \right] (u - x_1) + |u|^p, \quad x \in \Omega_{F^-}(u_1, -2\varepsilon),$$

and

(5.8)

$$F^-(x; u_1, \infty) = p\varepsilon^{p-1} e^{u/\varepsilon} \left[ \int_{u/\varepsilon}^{\infty} t^{p-1} e^{-t} dt \right] (x_1 - u) + u^p, \quad x \in \Omega_{F^-}(u_1, \infty), \quad 0 \leq u_1 \leq u,$$

with  $u$  given by

$$u = x_1 - \varepsilon + \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

We will also need  $\text{sgn}(F_{x_2 x_2}^-(x; u_1, \infty)) = \text{sgn}(\tau_-)$ , where, according to (4.12),

$$(5.9) \quad \tau_-(u) = -p(p-1)(p-2)\varepsilon^{p-2} \int_{u/\varepsilon}^{\infty} t^{p-3} e^{-t} dt.$$

We will now build global Bellman candidates starting “from the middle”: the only canonical sub-domains that can symmetrically incorporate the line  $x_1 = 0$  are  $\Omega_L(0)$  and  $\Omega_T(0)$ . We fix one of these and glue other canonical sub-domains to it, so as to preserve the global convexity/concavity of the resulting candidate. It is convenient to split further discussion in two parts:  $p \geq 1$  and  $0 < p < 1$ .

Before we proceed, let us fix the following notation for the coefficients of  $x_1$  in (5.1) and (5.7) for  $f(s) = |s|^p$ :

$$(5.10) \quad m_+(u; u_1) = e^{-u/\varepsilon} \left[ \frac{u_1^p - |u_1 - 2\varepsilon|^p}{2\varepsilon} e^{u_1/\varepsilon} + p\varepsilon^{p-1} \int_{u_1/\varepsilon}^{u/\varepsilon} t^{p-1} e^t dt \right], \quad 0 \leq u_1 \leq u,$$

$$(5.11) \quad m_-(u; u_2) = e^{u/\varepsilon} \left[ \frac{u_2^p - (u_2 + 2\varepsilon)^p}{2\varepsilon} e^{-u_2/\varepsilon} + p\varepsilon^{p-1} \int_{u/\varepsilon}^{u_2/\varepsilon} t^{p-1} e^{-t} dt \right], \quad 0 \leq u \leq u_2.$$

In addition, although most of our work here is with the power function, in Section 9 we will need the more general counterparts of (5.10) and (5.11) in the specific cases when  $u_1 = \varepsilon$  and  $u_2 = \infty$ , respectively:

$$(5.12) \quad m_+^f(u) = e^{-u/\varepsilon} \int_1^{u/\varepsilon} f'(\varepsilon t) e^t dt,$$

$$(5.13) \quad m_-^f(u) = e^{u/\varepsilon} \int_{u/\varepsilon}^{\infty} f'(\varepsilon t) e^{-t} dt.$$

5.1. **The case  $p \geq 1$ .** Let us first consider the split  $\Omega_\varepsilon = \Omega_{F^-}(-\infty, -\varepsilon) \cup \Omega_L(0) \cup \Omega_{F^+}(\varepsilon, \infty)$ . According to (4.13), the solution in  $\Omega_L(0)$  is given by

$$G(x) = L_0(x) = x_2^{p/2},$$

hence, in that region we have  $G_{x_2 x_2}(x) = \frac{1}{4}p(p-2)x^{p/2-2}$  and so  $\text{sgn}(G_{x_2 x_2}) = \text{sgn}(p-2)$ . In  $\Omega_{F^+}(\varepsilon, \infty)$ , by (5.4) we have  $\text{sgn}(F_{x_2 x_2}^+) = \text{sgn}(p-2)$  for  $p \geq 1$ . We, thus, attempt to check whether setting  $G(x) = F^+(x; \varepsilon, \infty)$  in  $\Omega_{F^+}(\varepsilon, \infty)$  will produce an acceptable Bellman candidate. Along the line  $x_2 = \varepsilon^2$  (the shared boundary of the two sub-domains), we have  $L_0 = \varepsilon^p$ , while (5.2) gives  $F^+ = \varepsilon^p$  for  $u = \varepsilon$ . In Section 6, we will verify the convexity/concavity of the resulting candidate in the combined domain  $\Omega_L(0) \cup \Omega_{F^+}(\varepsilon, \infty)$ . Subject to that verification, we have a complete candidate in the  $\Omega_\varepsilon^+$  and hence, by symmetry, in the whole  $\Omega_\varepsilon$ :

$$(5.14) \quad M(x) = \begin{cases} F^-(x; -\infty, -\varepsilon), & x \in \Omega_{F^-}(-\infty, -\varepsilon), \\ L_0(x), & x \in \Omega_L(0), \\ F^+(x; \varepsilon, \infty), & x \in \Omega_{F^+}(\varepsilon, \infty). \end{cases}$$

More explicitly,

$$(5.15) \quad M_{\varepsilon,p}(x) = \begin{cases} m_+(u; \varepsilon)(|x_1| - u) + u^p, & x \in \Omega_{F^-}(-\infty, -\varepsilon) \cup \Omega_{F^+}(\varepsilon, \infty), \\ x_2^{p/2}, & x \in \Omega_L(0), \end{cases}$$

with

$$u = |x_1| + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

The corresponding foliation of  $\Omega_\varepsilon$  is shown on Figure 6. This function gives an upper Bellman

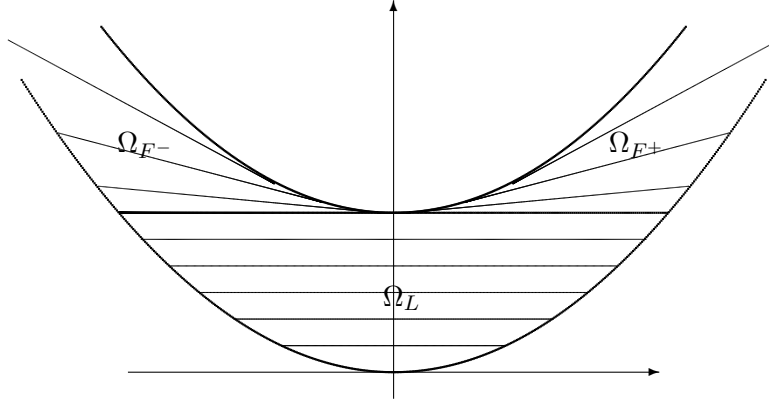


FIGURE 6. The Bellman foliation for the candidate  $M$ .

candidate for  $1 \leq p \leq 2$  and a lower one for  $2 \leq p < \infty$ . Let us write separately the candidate for the important case  $p = 1$ , when the integrals can be evaluated explicitly:

$$(5.16) \quad M_{\varepsilon,1}(x) = \begin{cases} |x_1| + \left( \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2} \right) \exp \frac{-|x_1| + \sqrt{\varepsilon^2 - x_2 + x_1^2}}{\varepsilon}, & x \in \Omega_{F^-}(-\infty, 0) \cup \Omega_{F^+}(0, \infty), \\ x_2^{1/2}, & x \in \Omega_L(0). \end{cases}$$

To get the other Bellman candidate, we consider the split  $\Omega_\varepsilon = \Omega_{F^+}(-\infty, 0) \cup \Omega_T(0) \cup \Omega_{F^-}(0, \infty)$ . In  $\Omega_{F^-}(0, \infty)$  we, naturally, set

$$G(x) = F^-(x; 0, \infty).$$

According to (5.9),  $\text{sgn}(F_{x_2 x_2}^-) = -\text{sgn}(p-2)$ . On the shared boundary of  $\Omega_T(0)$  and  $\Omega_{F^-}(0, \infty)$ , the line  $x_2 = 2\varepsilon x_1$ , (5.8) gives

$$F^-|_{u=0} = p\varepsilon^{p-1} \left[ \int_0^\infty t^p e^{-t} dt \right] x_1 = p\varepsilon^{p-1} \Gamma(p) x_1 = \frac{p}{2} \varepsilon^{p-2} \Gamma(p) x_2.$$

In  $\Omega_T(0)$ , we set

$$G(x) = T_0(x),$$

where, from (4.20),

$$T_0(x) = \alpha x_2.$$

To preserve continuity along the line  $x_2 = 2\varepsilon x_1$ , we set  $\alpha = \frac{p}{2} \varepsilon^{p-2} \Gamma(p)$ . Again, we postpone until the next section the verification that the resulting candidate is locally convex/concave in  $\Omega_T(0) \cup \Omega_{F^-}(0, \infty)$ . By symmetry, we obtain the following global candidate:

$$(5.17) \quad N(x) = \begin{cases} F^+(x; -\infty, 0), & x \in \Omega_{F^+}(-\infty, 0), \\ T_0(x), & x \in \Omega_T(0), \\ F^-(x; 0, \infty), & x \in \Omega_{F^-}(0, \infty). \end{cases}$$

More specifically,

$$(5.18) \quad N_{\varepsilon,p}(x) = \begin{cases} m_-(u; \infty)(|x_1| - u) + u^p, & x \in \Omega_{F^+}(-\infty, 0) \cup \Omega_{F^-}(0, \infty), \\ \frac{p}{2} \varepsilon^{p-2} \Gamma(p) x_2, & x \in \Omega_T(0), \end{cases}$$

where

$$u = |x_1| - \varepsilon + \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

This function gives an upper candidate for  $p \geq 2$  and a lower one for  $1 \leq p \leq 2$ . The corresponding Bellman foliation is shown on Figure 7. We note that this geometric description is accurate for all  $p > 1$ , but not for  $p = 1$ . In that case, we can again evaluate the integrals explicitly and thus obtain

$$(5.19) \quad N_{\varepsilon,1}(x) = \begin{cases} |x_1|, & x \in \Omega_{F^+}(-\infty, 0) \cup \Omega_{F^-}(0, \infty), \\ \frac{1}{2\varepsilon} x_2, & x \in \Omega_T(0). \end{cases}$$

Therefore,  $N_{\varepsilon,1}$  is a piecewise linear function and the defect of its Hessian is 2 in the interior of each canonical sub-domain involved. Thus every straight line lying entirely in  $\Omega_{F^-}(0, \infty)$ ,  $\Omega_{F^+}(-\infty, 0)$ , or  $\Omega_T(0)$  is an extremal trajectory for  $N_{\varepsilon,1}$ .

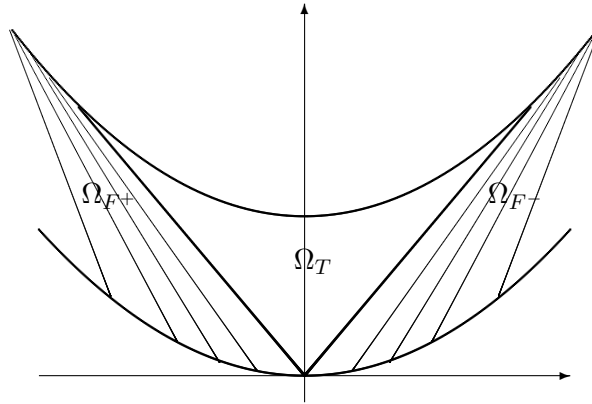


FIGURE 7. The Bellman foliation for the candidate  $N$ .

**5.2. The case  $0 < p < 1$ .** In this case, we have two Bellman candidates of a more complicated nature. As before, we build each global solution starting with either  $L_0(x)$  or  $T_0(x)$  (thus placing either  $\Omega_L(0)$  or  $\Omega_T(0)$  at the center of  $\Omega_\varepsilon$ ) and then extending the solution appropriately to the whole  $\Omega_\varepsilon$ .

Let us first set  $G(x) = L_0(x) = x_2^{p/2}$  in  $\Omega_L(0)$ . Since  $\text{sgn}(L_{x_2x_2}) = \text{sgn}(p(p-2)) < 0$ , we are building an upper Bellman candidate. It is natural to attempt to glue a solution  $F^+(x; \varepsilon, u_2)$  to this foundation, for some  $u_2 > 0$ , just as we did in the previous case. Since  $\text{sgn}(F_{x_2x_2}^+) = \text{sgn}(\tau_+)$ , and, by (5.4),  $\tau_+(\varepsilon) < 0$ , this is a proper choice in that we still have a concave candidate. However, we cannot take  $u_2 = \infty$ , as we did before, since  $\tau_+(u) > 0$  for sufficiently large  $u$ . This means that the only canonical solution we can have after all transient effects have dissipated (i.e. for large  $u$ , after all necessary transition regimes have been deployed) is  $F^-(x; u_1, \infty)$ . Again, we check the sign of  $F_{x_2x_2}^- : \text{sgn}(F_{x_2x_2}^-) = \text{sgn}(\tau_-)$  and, by (5.9),  $\tau_-(u) < 0$  for all  $u$ . Therefore, we need a transition regime connecting the foliations for  $F^+(x; \varepsilon, u_1)$  and  $F^-(x; u_1, \infty)$ . We have an obvious choice, one that was considered in section 4.4:  $T(x; \xi)$  for a specific value of  $\xi$ . Condition (4.26) dictates that  $T(x; \xi)$  would appropriately glue  $F^+(x; \varepsilon, \xi)$  and  $F^-(x; \xi, \infty)$  if and only if

$$(5.20) \quad m_+(\xi) + m_-(\xi) = 2p\xi^{p-1}.$$

Slightly rewriting the integrals in (5.10) and (5.11), and letting  $\xi = \mu\varepsilon$ , we can reformulate this condition as follows:

$$(5.21) \quad e^{-\mu} \int_1^\mu z^{p-1} e^z dz + e^\mu \int_\mu^\infty z^{p-1} e^{-z} dz = 2\mu^{p-1}.$$

Obviously, we need to have  $\mu > 1$ , that is  $\xi - \varepsilon = \varepsilon(\mu - 1) > 0$ . We must verify that such a  $\mu$  exists and that  $\tau_+(\xi) \leq 0$ , meaning  $F^+$  remains an upper candidate up to the line  $u = \xi$ . Changing the variable in (5.4) and integrating by parts twice, we rewrite this condition as

$$(5.22) \quad e^{-\mu} \int_1^\mu z^{p-1} e^z dz \leq \mu^{p-1} + (1-p)\mu^{p-2}.$$

We are about to prove the existence of the solution of (5.21) satisfying (5.22). We prove a slightly more general result, which we will need in Section 9. Namely, let us replace the condition (5.20) with

$$m_+^f(\xi) + m_-^f(\xi) = 2f'(\xi),$$

where the functions  $m_\pm^f$  are defined by (5.12) and (5.13). Rewriting the integrals, we get

$$(5.23) \quad e^{-\mu} \int_1^\mu f'(z\varepsilon) e^z dz + e^\mu \int_\mu^\infty f'(z\varepsilon) e^{-z} dz = 2f'(\mu\varepsilon).$$

In addition, recall that for a general  $f$  we have  $\text{sgn}(F_{x_2x_2}^+) = \text{sgn}(\tau_+^f)$ , where we have set, using  $u_1 = \varepsilon$  and  $C = 0$  in (4.8),

$$\tau_+^f(u) = (f''(u) - f'(u)/\varepsilon)e^{u/\varepsilon} + \frac{1}{\varepsilon^2} \int_\varepsilon^u f'(s)e^{s/\varepsilon} ds.$$

After rewriting, we get  $\text{sgn}(\tau_+^f(\xi)) = \text{sgn}(\tau_+^f(\varepsilon\mu)) = \text{sgn}(g(\mu))$ , where

$$g(\mu) \stackrel{\text{def}}{=} e^{-\mu} \int_1^\mu f'(z\varepsilon) e^z dz - f'(\mu\varepsilon) + \varepsilon f''(\mu\varepsilon).$$

**Lemma 5.1.** Fix  $\varepsilon > 0$  and let  $f$  be a thrice-differentiable function on  $(\varepsilon, \infty)$  satisfying

$$(5.24) \quad \lim_{s \rightarrow \infty} e^s |f'''(s\varepsilon)| = \infty$$

and either

$$\text{Case 1:} \quad f'(t) \geq 0, \quad f''(t) \leq 0, \quad f'''(t) \geq 0, \quad \forall t \in (\varepsilon, \infty),$$

or

$$\text{Case 2: } f'(t) \leq 0, f''(t) \geq 0, f'''(t) \leq 0, \quad \forall t \in (\varepsilon, \infty).$$

Then, for each  $\varepsilon > 0$ , equation (5.23) has a unique solution  $\mu^*$  in the interval  $(1, \infty)$ . Furthermore,  $\mu^*$  satisfies

$$(5.25) \quad \begin{aligned} g(\mu_*) &\leq 0 \quad \text{in Case 1,} \\ g(\mu_*) &\geq 0 \quad \text{in Case 2.} \end{aligned}$$

*Proof.* Letting

$$h(\mu) = \int_1^\mu f'(z\varepsilon)e^z dz + e^{2\mu} \int_\mu^\infty f'(z\varepsilon)e^{-z} dz - 2e^\mu f'(\mu\varepsilon),$$

we see that the task is to prove that  $h$  has a unique zero  $\mu^*$  in  $(1, \infty)$ . We calculate:

$$h'(\mu) = 2e^{2\mu} \left[ \int_\mu^\infty f'(z\varepsilon)e^{-z} dz - f'(\mu\varepsilon)e^{-\mu} - \varepsilon f''(\mu\varepsilon)e^{-\mu} \right] = 2\varepsilon^2 e^{2\mu} \int_\mu^\infty f'''(z\varepsilon)e^{-z} dz.$$

Thus,

– Case 1:  $h'(\mu) > 0$ ,  $\forall \mu \geq 1$ , and, from (5.24),  $\lim_{\mu \rightarrow \infty} h'(\mu) = \infty$ . In addition,

$$h(1) = e^2 \int_1^\infty f'(z\varepsilon)e^{-z} dz - 2ef'(\varepsilon) = \varepsilon e^2 \int_1^\infty f''(z\varepsilon)e^{-z} dz - f'(\varepsilon)e < 0.$$

– Case 2:  $h'(\mu) < 0$ ,  $\forall \mu \geq 1$ ,  $\lim_{\mu \rightarrow \infty} h'(\mu) = -\infty$ , and  $h(1) > 0$ .

In each case, this implies the existence of a unique root  $\mu_*$  of  $h$ .

To finish the proof, observe that

$$e^{-\mu_*} \int_1^{\mu_*} f'(z\varepsilon)e^z dz = 2f'(\mu_*\varepsilon) - e^{\mu_*} \int_{\mu_*}^\infty f'(z\varepsilon)e^{-z} dz$$

and so we have

$$g(\mu_*) = -e^{\mu_*} \int_{\mu_*}^\infty f'(z\varepsilon)e^{-z} dz + f'(\mu_*\varepsilon) + \varepsilon f''(\mu_*\varepsilon) = -\varepsilon e^{\mu_*} \int_{\mu_*}^\infty f'''(z\varepsilon)e^{-z} dz,$$

which yields (5.25). □

Setting  $f(s) = |s|^p$ , we obtain an immediate

**Corollary 5.2.** *For each  $p < 1$ , equation (5.21) has a unique solution  $\mu_*$  in the interval  $(1, \infty)$ . Furthermore,  $\mu_*$  satisfies (5.22).*

*Remark 5.3.* It is easy to show that  $\mu_*(p) \rightarrow \infty$ , as  $p \rightarrow 1^-$  and  $\mu_*(p) \rightarrow 1$ , as  $p \rightarrow -\infty$ .

From now on, let us denote the solution of (5.21) simply by  $\mu$ ; also let  $\xi = \mu\varepsilon$ . The lemma just proved means that we have, indeed, succeeded in building a complete Bellman candidate. On  $\Omega_T(\xi)$  that candidate is given by  $T(x; \xi)$ , where, according to (4.27),

$$T(x; \xi) = p\xi^{p-1}x_1 + \frac{1}{2\varepsilon}(m_-(\xi; \infty) - p\xi^{p-1})(-2x_1\xi + x_2 + \xi^2) + (1-p)\xi^p.$$

Extending, as before, the solution to the left of the line  $x_1 = 0$  by symmetry, we can write down our global candidate:

$$(5.26) \quad P(x) = \begin{cases} F^+(x; -\infty, -\xi), & x \in \Omega_{F^+}(-\infty, -\xi), \\ T(x; -\xi), & x \in \Omega_T(-\xi), \\ F^-(x; -\xi, -\varepsilon), & x \in \Omega_{F^-}(-\xi, -\varepsilon), \\ L_0(x), & x \in \Omega_L(0), \\ F^+(x; \varepsilon, \xi), & x \in \Omega_{F^+}(\varepsilon, \xi), \\ T(x; \xi), & x \in \Omega_T(\xi), \\ F^-(x; \xi, \infty), & x \in \Omega_{F^-}(\xi, \infty). \end{cases}$$

This representation exhibits the geometric structure of  $P$ . In addition, we need a usable formula:

$$(5.27) \quad P_{\varepsilon,p}(x) = \begin{cases} m_-(u_-; \infty)(|x_1| - u_-) + u_-^p, & x \in \Omega_{F^+}(-\infty, -\xi) \cup \Omega_{F^-}(\xi, \infty), \\ p\xi^{p-1}|x_1| + \frac{1}{2\varepsilon}(m_-(\xi; \infty) - p\xi^{p-1})(-2|x_1|\xi + x_2 + \xi^2) + (1-p)\xi^p, & x \in \Omega_T(-\xi) \cup \Omega_T(\xi), \\ m_+(u_+; \varepsilon)(|x_1| - u_+) + u_+^p, & x \in \Omega_{F^-}(-\xi, -\varepsilon) \cup \Omega_{F^+}(\varepsilon, \xi), \\ x_2^{p/2}, & x \in \Omega_L(0), \end{cases}$$

where  $\xi = \mu\varepsilon$ ,  $\mu$  is the unique solution of (5.21) in  $(1, \infty)$ , and

$$u_+ = |x_1| + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}, \quad u_- = |x_1| - \varepsilon + \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

As noted before,  $P_{\varepsilon,p}$  gives an upper Bellman candidate for  $0 < p < 1$ . The foliation for this function is shown on Figure 8.

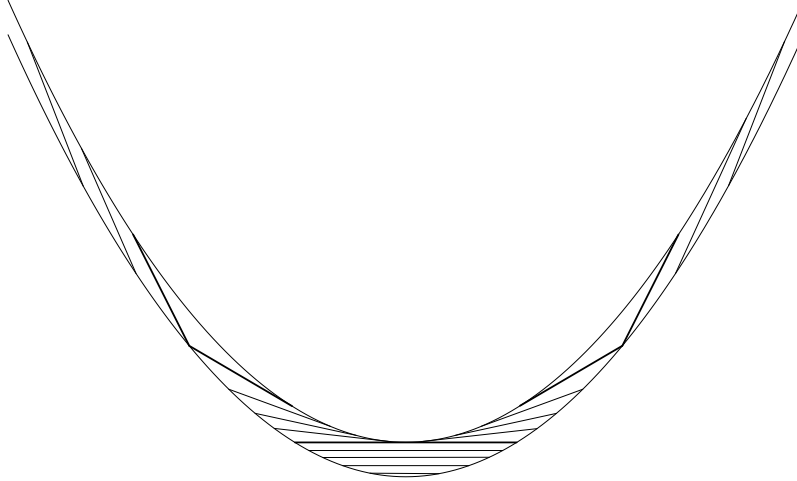


FIGURE 8. The Bellman foliation for the candidate  $P$ .

To construct the lower candidate, we place  $\Omega_T(0)$  at the center of  $\Omega_\varepsilon$ , thus setting

$$(5.28) \quad G(x) = T_0(x) = \alpha x_2, \quad x \in \Omega_T(0),$$

with the constant  $\alpha$  still to be determined. We now have to glue a canonical candidate to the right of  $T_0(x)$ . Geometrically, we need a candidate whose foliation includes the line  $x_2 = 2\varepsilon x_1$  and, thus, have three choices:  $F^-(x; 0, \beta)$ , for some  $\beta$ ;  $L^+(x; \varepsilon)$ ; and  $L^-(x; \varepsilon)$ .

Let us first examine  $F^-(x; 0, \beta)$ . We have  $\text{sgn}(F_{x_2 x_2}^-) = \text{sgn}(\tau_-)$ , where, according to (4.11),  $\tau_-(u) = p(p-1)O(u^{p-2})$  for small positive  $u$ . This means that near the boundary of  $\Omega_T(0)$  (i.e. the line  $u = 0$ ), we have  $F_{x_2 x_2}^- < 0$ , that is  $F^-$  gives an *upper* candidate, while we are building a lower one.

For the other two possibilities, (4.19) gives

$$\text{sgn}(L_{x_2 x_2}^\pm) = \mp \text{sgn}(p(p-1)(p-2)),$$

meaning only  $L^-$  gives a lower candidate. Therefore, we set

$$G(x) = L^-(x; \varepsilon) = x_2^{p-1} x_1^{2-p}, \quad x \in \Omega_L(\varepsilon),$$

where we have used the “ $-$ ” part of formula (4.17) with  $u_- = 0$  and  $f(s) = |s|^p$ .

We are now in a position to determine the constant  $\alpha$  in (5.28): setting  $T_0(x_1, 2\varepsilon x_1) = L^-(x_1, 2\varepsilon x_1)$  gives

$$\alpha = (2\varepsilon)^{p-2}.$$

Having determined our candidate in  $\Omega_T(0) \cup \Omega_L(\varepsilon)$ , we now have to glue another canonical solution to  $L^-$ . Observe that for sufficiently large  $x_1$  we expect our candidate to be given by  $F^+(x; \gamma, \infty)$  (according to (5.9), its counterpart,  $F^-$ , determines an upper candidate and so does not work here). We attempt to take  $\gamma = 2\varepsilon$ , i.e. glue  $F^+$  directly to  $L^-$  without further transition regimes. From (5.5),  $\text{sgn}(\tau_+(u)) > 0$ ,  $\forall u \geq 2\varepsilon$ . Therefore, we have obtained the following complete lower candidate:

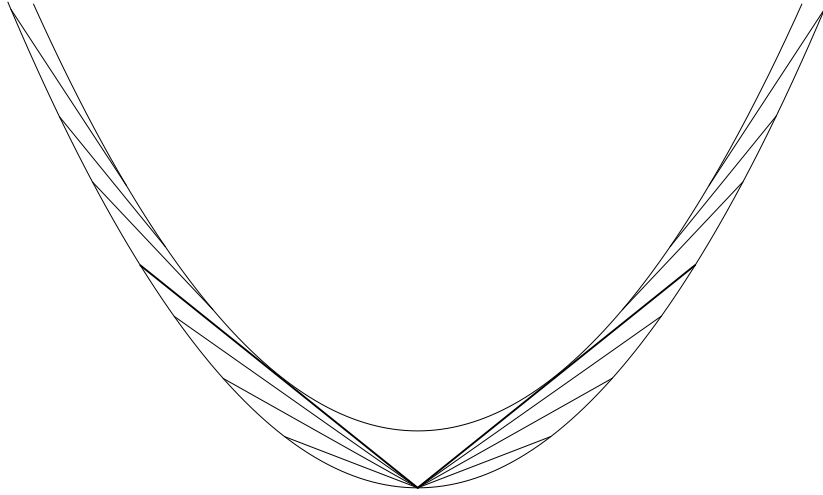


FIGURE 9. The Bellman foliation for the candidate  $R$ .

$$(5.29) \quad R(x) = \begin{cases} F^-(x; -\infty, -2\varepsilon), & x \in \Omega_{F^-}(-\infty, -2\varepsilon), \\ L^+(x; -\varepsilon), & x \in \Omega_L(-\varepsilon), \\ T_0(x), & x \in \Omega_T(0), \\ L^-(x; \varepsilon), & x \in \Omega_L(\varepsilon), \\ F^+(x; 2\varepsilon, \infty), & x \in \Omega_{F^+}(2\varepsilon, \infty). \end{cases}$$

Written explicitly, the function  $R$  is given by

$$(5.30) \quad R_{\varepsilon,p}(x) = \begin{cases} m_+(u; 2\varepsilon)(|x_1| - u) + u^p, & x \in \Omega_{F^-}(-\infty, -2\varepsilon) \cup \Omega_{F^+}(2\varepsilon, \infty), \\ x_2^{p-1} |x_1|^{2-p}, & x \in \Omega_L(-\varepsilon) \cup \Omega_L(\varepsilon), \\ (2\varepsilon)^{p-2} x_2, & x \in \Omega_T(0), \end{cases}$$

where

$$u = |x_1| + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

The corresponding foliation is shown on Figure 9.

## 6. BELLMAN INDUCTION

In this section, we first establish the local concavity/convexity properties of the global Bellman candidates in the whole domain  $\Omega_\varepsilon$ , i.e. show that each candidate  $G$  satisfies either

$$(6.1) \quad G(\alpha_- x^- + \alpha_+ x^+) \geq \alpha_- G(x^-) + \alpha_+ G(x^+)$$

or

$$(6.2) \quad G(\alpha_- x^- + \alpha_+ x^+) \leq \alpha_- G(x^-) + \alpha_+ G(x^+)$$

for all non-negative numbers  $\alpha_\pm$  such that  $\alpha_- + \alpha_+ = 1$  and all  $x^-, x^+ \in \Omega_\varepsilon$  such that the entire line segment  $[x^-, x^+]$  is inside  $\Omega_\varepsilon$ . With this in hand, we then use induction on scales to show that each candidate appropriately majorates (minorates) the Bellman function for which it was constructed.

Before verifying (6.1) or (6.2) for our Bellman candidates, we need to make two observations. First, note that we can consider (6.1) or (6.2) as the statement that the derivative of  $G$  along every direction is decreasing (respectively, increasing) in that direction; thus, each property can be checked locally. We know from Section 5 that each global candidate  $G$  has the Hessian of the appropriate sign — and so the required monotonicity of the derivatives — in every canonical subdomain. Therefore, the only places where (6.1) or (6.2) needs to be verified are the points where the segment  $[x^-, x^+]$  intersects the boundaries between subdomains. For each such boundary, that verification will take the form of measuring the jump in the derivative along any direction transversal to the boundary. While in general our global candidates  $G$  are not guaranteed to be smooth at such points, we do have one-sided transversal derivatives everywhere and can check the sign of that jump. If that jump turns out to be 0 at all points of the boundary, that means that the two solutions are, in fact, glued  $C^1$ -smoothly. In this case, we will automatically obtain a global candidate satisfying either (6.1) or (6.2) in the union of the two subdomains.

The second important observation is the following: whenever we glue two subdomains, it is always along a line that is an extremal trajectory for both local foliations. For this reason, while technically we need to check the sign of the jump in the derivative at all points of the shared boundary and in all directions transversal to it, it suffices to do so at a single boundary point, in a single transversal direction. Let us explain: recall that for any solution  $G$  of the Monge–Ampère equation, the gradient  $(G_{x_1}, G_{x_2}) = (t_1, t_2)$  is constant along any extremal line; hence, the jump in the vector  $(t_1, t_2)$  is constant along the boundary and we can measure it at any point. On the other hand, consider new coordinates  $(y_1, y_2)$ , where  $y_1$  is directed along the shared boundary and  $y_2$ , in a fixed transversal direction. Then  $G_{y_1}$  is continuous (so no jump in that component), because both solutions are linear functions along  $y_1$  and they were glued continuously. Thus, we only need to measure the jump in  $G_{y_2}$ .

We are now in a position to prove the following

### Lemma 6.1.

- (1) The function  $M_{\varepsilon,p}$  given by (5.15) satisfies (6.1) for  $1 \leq p \leq 2$  and (6.2) for  $p \geq 2$ .
- (2) The function  $N_{\varepsilon,p}$  given by (5.18) satisfies (6.2) for  $1 \leq p \leq 2$  and (6.1) for  $p \geq 2$ .
- (3) The function  $P_{\varepsilon,p}$  given by (5.27) satisfies (6.1) for  $0 < p < 1$ .
- (4) The function  $R_{\varepsilon,p}$  given by (5.30) satisfies (6.2) for  $0 < p \leq 1$ .

*Proof.* According to the preceding discussion, we need to check all boundaries between subdomains for each candidate. Since such boundaries are never parallel to the  $x_2$ -axis, we can



choose  $x_2$  as our transversal direction in all cases. Therefore, to prove the lemma, we will use the following procedure: for each specific candidate  $G$  and each boundary between two of its subdomains, pick one point  $x = (x_1, x_2)$  on the boundary and verify that  $t_2(x^+) \leq t_2(x^-)$  for (6.1) and  $t_2(x^+) \geq t_2(x^-)$  for (6.2), where  $t_2(x^\pm) = G_{x_2}(x_1, x_2 \pm 0)$ .

In all cases, by symmetry it is sufficient to check only those subdomain boundaries that are in  $\Omega_\varepsilon^+$ .

- (1) In the case of  $M_{\varepsilon,p}$ , we check the jump in  $t_2$  along the line  $x_2 = \varepsilon^2$  separating  $\Omega_L(0)$  and  $\Omega_{F+}(\varepsilon, \infty)$ . From (4.4) and (4.5), we have, for any point  $x$  on this line,

$$t_2(x^+) = \frac{1}{2}m'(\varepsilon) = \frac{1}{2\varepsilon}(p\varepsilon^{p-1} - m(\varepsilon)),$$

where  $m(\varepsilon) = 0$  from (5.10) with  $u = u_1 = \varepsilon$ . Thus, we have  $t_2(x^+) = \frac{p}{2}\varepsilon^{p-2}$ .

On the other hand, from (5.15)  $t_2(x^-) = \frac{p}{2}x_2^{p/2-1} = \frac{p}{2}(\varepsilon^2)^{p/2-1} = \frac{p}{2}\varepsilon^{p-2}$ . Therefore, the derivative jump is zero for any  $p$  and the first statement of the lemma is proved.

- (2) For  $N_{\varepsilon,p}$ , we check the jump in  $t_2$  along the line  $x_2 = 2\varepsilon x_1$  separating  $\Omega_T(0)$  and  $\Omega_{F-}(0, \infty)$ . From (4.4) and (4.10), we have, for any point  $x$  on this line,

$$t_2(x^-) = \frac{1}{2}m'(0) = \frac{1}{2\varepsilon}m(0),$$

where  $m(0) = p\varepsilon^{p-1} \int_0^\infty t^{p-1} e^{-t} dt = p\varepsilon^{p-1}\Gamma(p)$  from (5.11) with  $u = 0$  and  $u_2 = \infty$ . Thus, we have  $t_2(x^-) = \frac{p}{2}\varepsilon^{p-2}\Gamma(p)$ .

On the other hand, from (5.18)  $t_2(x^+) = \frac{p}{2}\varepsilon^{p-2}\Gamma(p)$ . Again, the derivative jump is zero and the second statement is proved.

- (3) For  $P_{\varepsilon,p}$ , we need to check three boundary lines:  $x_2 = \varepsilon^2$  between  $\Omega_L(0)$  and  $\Omega_{F+}(\varepsilon, \xi)$ ;  $x_2 = 2(\xi + \varepsilon)x_1 - 2\xi\varepsilon - \xi^2$  between  $\Omega_{F+}(\varepsilon, \xi)$  and  $\Omega_T(\xi)$ ; and  $x_2 = 2(\xi - \varepsilon)x_1 + 2\xi\varepsilon - \xi^2$  between  $\Omega_T(\xi)$  and  $\Omega_{F-}(\xi, \infty)$ . The first verification is the same as in part (1) above. The second and third are automatic: the value of  $\xi = \mu\varepsilon$  in (5.27) was chosen according to Corollary 5.2, which ensured that condition (5.20) is satisfied. That condition, in turn, was a criterion for having zero jump in  $t_2$  across each bounding tangent of  $\Omega_T(\xi)$ .
- (4) For  $R_{\varepsilon,p}$ , we have two segments of the same line to check: for  $0 \leq x_1 \leq \varepsilon$ , the line  $x_2 = 2\varepsilon x_1$  separates  $\Omega_T(0)$  and  $\Omega_L(\varepsilon)$ ; for  $\varepsilon \leq x_1 \leq 2\varepsilon$ , the same line separates  $\Omega_L(\varepsilon)$  and  $\Omega_{F+}(2\varepsilon, \infty)$ .

In the first case,  $t_2(x^+) = (2\varepsilon)^{p-2}$  and  $t_2(x^-) = (p-1)x_2^{p-2}x_1^{2-p}$  from (5.30). When  $x_2 = 2\varepsilon x_1$ , we have  $t_2(x^-) = (p-1)(2\varepsilon)^{p-2}$  and so  $t_2(x^+) - t_2(x^-) = (2\varepsilon)^{p-2}(2-p) > 0$ , which is consistent with  $R_{\varepsilon,p}$  being a lower Bellman candidate.

In the second case,  $t_2(x^+)$  is given, similarly to part (1), by  $t_2(x^+) = \frac{1}{2}m'(2\varepsilon) = \frac{1}{2\varepsilon}(p(2\varepsilon)^{p-1} - m(2\varepsilon))$ , where  $m(2\varepsilon) = (2\varepsilon)^{p-1}$  from (5.10) with  $u = u_1 = 2\varepsilon$ . Thus,  $t_2(x^+) = (p-1)(2\varepsilon)^{p-2}$ . On the other hand,  $t_2(x^-) = (p-1)(2\varepsilon)^{p-2}$ , as before, and again we have  $C^1$  smoothness of  $R_{\varepsilon,p}$ .

□

We will make use of the following geometric result, whose proof can be found in [SV].

**Lemma 6.2.** *Fix  $\varepsilon > 0$ . Take any  $\delta > \varepsilon$ . Then for every interval  $I$  and every  $\varphi \in \text{BMO}_\varepsilon(I)$ , there exists a splitting  $I = I_- \cup I_+$  such that the whole straight-line segment with the endpoints  $x^\pm = (\langle \varphi \rangle_{I_\pm}, \langle \varphi^2 \rangle_{I_\pm})$  is inside  $\Omega_\delta$ . Moreover, the splitting parameter  $\alpha_+ = |I_+|/|I|$  can be chosen uniformly (with respect to  $\varphi$  and  $I$ ) separated from 0 and 1.*

We need another simple lemma that says that cutting a BMO function off at a given height does not increase its norm, which is implicitly contained in [SV] as well.

**Lemma 6.3.** Fix  $\varphi \in \text{BMO}(\mathbb{R}^n)$  and  $c, d \in \mathbb{R}$  such that  $c < d$ . Let  $\varphi_{c,d}$  be the cut-off of  $\varphi$  at heights  $c$  and  $d$ :

$$(6.3) \quad \varphi_{c,d}(s) = \begin{cases} c, & \text{if } \varphi(s) \leq c; \\ \varphi(s), & \text{if } c < \varphi(s) < d; \\ d, & \text{if } \varphi(s) \geq d. \end{cases}$$

Then

$$\langle \varphi_{c,d}^2 \rangle_J - \langle \varphi_{c,d} \rangle_J^2 \leq \langle \varphi^2 \rangle_J - \langle \varphi \rangle_J^2, \quad \forall \text{ cube } J,$$

and, consequently,

$$\|\varphi_{c,d}\|_{\text{BMO}} \leq \|\varphi\|_{\text{BMO}}.$$

*Proof.* First, let us note that it is sufficient to prove this lemma for a one-sided cut, for example, for  $c = -\infty$ . We then get the full statement by applying this argument twice. Indeed, if we denote by  $C_d\varphi$  the cut-off of  $\varphi$  from above at height  $d$ , i.e.  $C_d\varphi = \varphi_{-\infty,d}$ , then  $\varphi_{c,d} = -C_{-c}(-C_d\varphi)$ .

Take a cube  $J$  and let  $J_1 = \{s \in J: \varphi(s) \leq d\}$  and  $J_2 = \{s \in J: \varphi(s) > d\}$ . If either  $J_1 = \emptyset$  or  $J_2 = \emptyset$ , the statement is trivial. Thus, we may assume that  $J_k \neq \emptyset$ . Let  $\beta_k = |J_k|/|J|$ ,  $k = 1, 2$ . We have the following identity:

$$\begin{aligned} & [\langle \varphi^2 \rangle_J - \langle \varphi \rangle_J^2] - [\langle (C_d\varphi)^2 \rangle_J - \langle C_d\varphi \rangle_J^2] \\ &= \beta_2 [\langle \varphi^2 \rangle_{J_2} - \langle \varphi \rangle_{J_2}^2] + \beta_1 \beta_2 [\langle \varphi \rangle_{J_2} - d] [\langle \varphi \rangle_{J_2} + d - 2\langle \varphi \rangle_{J_1}], \end{aligned}$$

which proves the lemma, because  $\langle \varphi \rangle_{J_1} \leq d \leq \langle \varphi \rangle_{J_2}$ .  $\square$

The following is the main result of this section. Its statement is similar to — if much more general than — that of Lemma 2c from [SV]. The proof, by induction on pseudo-dyadic scales, is somewhat streamlined compared to that in [SV], although its main ingredients are the same.

**Lemma 6.4.** Fix  $\varepsilon > 0$  and let  $B$  and  $b$  be two functions defined and continuous on  $\Omega_\delta$  for some  $\delta > \varepsilon$ . Assume that  $B$  has property (6.1) and  $b$  has property (6.2) on  $\Omega_\delta$ . Let  $W(t) = B(t, t^2)$ ,  $w(t) = b(t, t^2)$ . If either  $W$  or  $w$  is unbounded at  $-\infty$  or  $+\infty$ , assume it is monotone for  $t$  sufficiently close to  $-\infty$  or  $+\infty$ , respectively.

Fix a point  $x \in \Omega_\varepsilon$  and an interval  $Q$  and take any function  $\varphi \in \text{BMO}_\varepsilon(Q)$  such that  $(\langle \varphi \rangle_Q, \langle \varphi^2 \rangle_Q) = x$ . Then

$$B(x) \geq \langle W(\varphi) \rangle_Q,$$

$$b(x) \leq \langle w(\varphi) \rangle_Q,$$

including the possible infinite values on either side of each inequality.

*Proof.* We will only prove the statement of the lemma concerning  $B$ , as the part concerning  $b$  is virtually identical. We first establish the result for those  $\varphi \in \text{BMO}_\varepsilon(Q)$  that are bounded and then approximate arbitrary BMO functions by appropriately chosen cut-offs.

Take  $\varphi \in \text{BMO}_\varepsilon(Q) \cap L^\infty(Q)$ . Observe that  $\varphi \in \text{BMO}_\varepsilon(I)$  for any subinterval  $I$  of  $Q$ . We now build a binary tree  $D(Q)$  of subintervals of  $Q$ , where every interval  $I \in D(Q)$  is split into two subintervals  $I_\pm \in D(Q)$  according to the rule from Lemma 6.2. The set of intervals of the  $n$ -th generation will be denoted by  $D_n(Q)$ , so  $D_0(Q) = \{Q\}$ ,  $D_1(Q) = \{Q_\pm\}$ , etc. For every interval  $I \in D(Q)$ , let  $x^I \in \Omega_\varepsilon$  be the corresponding Bellman point,  $x^I = (\langle \varphi \rangle_I, \langle \varphi^2 \rangle_I)$ . Let  $x^{(n)}$  denote the step function from  $Q$  into  $\Omega_\varepsilon$ , defined by the rule  $x^{(n)}(t) = x^I$  if  $t \in I$ ,  $I \in D_n(Q)$ . Since Lemma 6.2 provides for the value of  $\alpha_+$  uniformly separated from 0 and 1 on every step, we have

$$\max_{I \in D_n(Q)} \{|I|\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the Lebesgue differentiation theorem, we have  $x^{(n)}(t) \rightarrow (\varphi(t), \varphi^2(t))$  almost everywhere. Since  $\varphi$  is assumed bounded,  $\{x^{(n)}\}$  is a sequence of bounded functions.

For each of the splits prescribed by Lemma 6.2, the line segment connecting  $x^{I-}$ ,  $x^I$ , and  $x^{I+}$  lies in  $\Omega_\delta$ . Using the property (6.1) of  $B$  repeatedly, we get, for any  $n \geq 1$ ,

$$\begin{aligned} |Q|B(x^Q) &\geq |Q_+|B(x^{Q_+}) + |Q_-|B(x^{Q_-}) \\ &\geq \sum_{I \in D_n(Q)} |I|B(x^I) = \int_Q B(x^{(n)}(t)) dt. \end{aligned}$$

Since  $B$  is continuous on  $\Omega_\delta$  (and thus on  $\Omega_\varepsilon$ ), the dominated convergence theorem applies and taking the limit as  $n \rightarrow \infty$  proves the lemma for bounded  $\varphi$ .

Take now an arbitrary  $\varphi \in \text{BMO}_\varepsilon(Q)$ . For  $c, d \in \mathbb{R}$  such that  $c < d$ , let  $\varphi_{c,d}$  be defined by (6.3). We have  $\varphi_{c,d} \in L^\infty(Q)$  and, by Lemma 6.3,  $\|\varphi_{c,d}\|_{\text{BMO}} \leq \varepsilon$ . Therefore,

$$B(\langle \varphi_{c,d} \rangle_Q, \langle \varphi_{c,d}^2 \rangle_Q) \geq \langle W(\varphi_{c,d}) \rangle_Q.$$

We now take the limit in this inequality as  $c \rightarrow -\infty$ . Since  $B$  is continuous, the limit of the left-hand side is  $B(\langle \varphi_{-\infty,d} \rangle_Q, \langle \varphi_{-\infty,d}^2 \rangle_Q)$ , where  $\varphi_{-\infty,d} \stackrel{\text{def}}{=} d\chi_{\{\varphi \geq d\}} + \varphi\chi_{\{\varphi < d\}}$ . On the other hand, by the monotone convergence theorem, the limit of the right-hand side is  $\langle W(\varphi_{-\infty,d}) \rangle_Q$ . The same argument works if we let  $d \rightarrow \infty$ , which completes the proof.  $\square$

As an immediate corollary, we obtain the following

**Theorem 6.5.** *For any  $x \in \Omega_\varepsilon$ , we have*

$$\begin{aligned} \text{For } p \geq 2 : & \quad N_{\varepsilon,p}(x) \geq \mathbf{B}_{\varepsilon,p}(x), & M_{\varepsilon,p}(x) &\leq \mathbf{b}_{\varepsilon,p}(x), \\ \text{For } 1 \leq p < 2 : & \quad M_{\varepsilon,p}(x) \geq \mathbf{B}_{\varepsilon,p}(x), & N_{\varepsilon,p}(x) &\leq \mathbf{b}_{\varepsilon,p}(x), \\ \text{For } 0 < p < 1 : & \quad P_{\varepsilon,p}(x) \geq \mathbf{B}_{\varepsilon,p}(x), & R_{\varepsilon,p}(x) &\leq \mathbf{b}_{\varepsilon,p}(x). \end{aligned}$$

*Proof.* Let  $B_\varepsilon$  stand for any of the upper candidates in the statement of the theorem and  $b_\varepsilon$  for any of the lower candidates. Since each  $B_\varepsilon$  and each  $b_\varepsilon$  is continuous in  $\varepsilon$ , it is sufficient to prove that

$$B_\delta(x) \geq \mathbf{B}_{\varepsilon,p}(x), \quad b_\delta(x) \leq \mathbf{b}_{\varepsilon,p}(x), \quad \forall x \in \Omega_\varepsilon,$$

for all  $\delta > \varepsilon$ , and then let  $\delta \rightarrow \varepsilon$ .

Observe that each  $B_\delta$  and each  $b_\delta$  satisfies the conditions of Lemma 6.4 and  $B_\delta(t, t^2) = b_\delta(t, t^2) = |t|^p$ . Therefore,

$$B_\delta(x) \geq \langle |\varphi|^p \rangle_Q, \quad b_\delta(x) \leq \langle |\varphi|^p \rangle_Q,$$

for all  $\varphi \in \text{BMO}_\varepsilon(Q)$  with  $(\langle \varphi \rangle_Q, \langle \varphi^2 \rangle_Q) = x$ . Now, take the supremum over all such  $\varphi$  in the first inequality and infimum in the second.  $\square$

## 7. OPTIMIZERS AND CONVERSE INEQUALITIES

In this section we construct, for each Bellman candidate  $G$  built in Section 5 and each point  $x \in \Omega_\varepsilon$ , a test function  $\varphi_x$  on  $(0, 1)$  with the following three properties:

$$\begin{aligned} (7.1) \quad (a) \quad & \langle \varphi_x \rangle_{(0,1)} = x_1, \quad \langle \varphi_x^2 \rangle_{(0,1)} = x_2; \\ (b) \quad & \varphi_x \in \text{BMO}_\varepsilon((0, 1)); \\ (c) \quad & \langle f(\varphi_x) \rangle_{(0,1)} = G(x). \end{aligned}$$

We will call each such function  $\varphi_x$  an *optimizer* for  $G(x)$ . We will often need optimizers for points on the top boundary of  $\Omega_\varepsilon$ , the parabola  $x_2 = x_1^2 + \varepsilon^2$ . It is convenient to parametrize these by the horizontal coordinate: let us denote  $\varphi_{(x_1, x_1^2 + \varepsilon^2)}$  by  $\psi_{x_1}$ . For the rest of this section,

we will simply write  $\text{BMO}_\varepsilon$  for  $\text{BMO}_\varepsilon((0, 1))$ . If we need  $\text{BMO}$  over another interval, we will write the interval explicitly.

If the candidate  $G$  corresponds to the Bellman function  $\mathbf{G}$ , the existence of  $\varphi$  satisfying (7.1) would immediately imply that

$$(7.2) \quad \begin{aligned} G(x) &\leq \mathbf{G}(x), \quad \forall x \in \Omega_\varepsilon, \quad \text{if } \mathbf{G} \text{ is an upper Bellman function,} \\ G(x) &\geq \mathbf{G}(x), \quad \forall x \in \Omega_\varepsilon, \quad \text{if } \mathbf{G} \text{ is a lower Bellman function.} \end{aligned}$$

In general, whether an extremal function exists for a particular inequality is a deep question, which sometimes is more difficult to answer than to prove the inequality itself. However, if the Bellman function for the inequality is known, one has a definitive answer as to the existence and the nature of optimizers. If one exists, it is found using the geometry of the Bellman foliation, as done in this section. If one does not exist (which is the case, for example, for all known Bellman functions for the maximal operator), optimizing sequences are built along the trajectories of the foliation.

Since our global Bellman candidates are built out of four canonical local blocks,  $F^+$ ,  $F^-$ ,  $L$ , and/or  $T$ , each with its own geometry, it is natural to attempt to construct a corresponding set of canonical optimizers, one for each local candidate. If one has such a set, one can demonstrate that (the appropriate line of) (7.2) holds for the global candidate  $G$  simply by showing it for each canonical sub-domain of  $\Omega_\varepsilon$ .

Thus, we fix a canonical block  $D$  and attempt to build an optimizer  $\varphi_x$  for each point of the domain  $\Omega_D$ . Expectedly,  $\varphi_x$  will, generally, depend not only on  $x$ , but also on the placement of  $\Omega_D$  within  $\Omega_\varepsilon$ . This is so because our local candidates, for the most part, themselves explicitly depend on that placement. Somewhat more subtly, the optimizers turn out to depend, in some cases, on exactly how the local candidates are glued together to produce a global one. In Section 4, we built blocks of two kinds. Some were completely determined by the parameters of their domain, namely  $F^+(x; -\infty, u_2)$ ,  $F^-(x; u_1, \infty)$ , and all blocks of the  $L$  type; let us call these blocks *complete*. Others, namely  $F^+(x; u_1, u_2)$  for  $u_1 > -\infty$ ,  $F^-(x; u_1, u_2)$  for  $u_2 < \infty$ , and  $T$ , had undetermined constants that were only found in Section 5, where the neighbors of these blocks in the global context were examined; let us call such blocks *incomplete*. Furthermore, different incomplete blocks may or may not require the knowledge of both neighbors. Thus  $F^+$  is *left-incomplete*, unless  $u_1 = -\infty$ , since it requires the knowledge of its left neighbor (which, from among our limited supply, can only be an  $L$  block);  $F^+$  is *right-incomplete*, unless  $u_2 = \infty$ , as it requires the neighbor on the right (again, an  $L$  block); and  $T$  is both left- and right-incomplete, as it requires both neighbors (these can be various combinations of  $L$  and  $F$  blocks).

This division has exact parallels in this section: to determine an optimizer for a complete block, we only need to know its domain. For example, it is meaningful to say that a given function is an optimizer for  $L^+(x; a)$ . However, to determine an optimizer for an incomplete block, we need to know its neighbors, and their optimizers, on the left and/or on the right, all the way to a complete block. Thus, we cannot say that a given function  $\varphi_x$  is an optimizer for  $F^+(x; u_1, u_2)$ , because  $F^+$  is incomplete. We could, however, say that a function is an optimizer for the sequence  $L(x; u_1 - \varepsilon) \rightarrow F^+(x; u_1, u_2)$ , because such a sequence determines the candidate completely.

In such situations, we will talk about *optimizers for a block* (alternatively, *local candidate in the context of a sequence*). Rather than defining this notion in general, let us list the specific sequences we will encounter, including those consisting of a single block:  $L$ ,  $F^+(x; -\infty, u_2)$ ,  $F^-(x; u_1, \infty)$ ,  $L \rightarrow F^+$ ,  $F^- \leftarrow L$ ,  $L \rightarrow T \leftarrow L$ ,  $F^+ \rightarrow T \leftarrow F^-$ , and  $L \rightarrow F^+ \rightarrow T \leftarrow F^-(x; u_1, \infty)$ . In each case, all arrows point to the block for which an optimizer is being

sought. Expectedly, each such sequence starts with a block that is complete on the left and ends with one complete on the right.

The main empirical consideration in building optimizers for Bellman foliations is that for each point  $x$  the whole construction of the optimizer  $\varphi_x$  should be taking place along the extremal trajectory passing through  $x$ . This means that  $\varphi_x$  should be such that when the interval  $I = (0, 1)$  is split into two subintervals,  $I = I_- \cup I_+$ , the Bellman points  $x^{I\pm} = (\langle \varphi_x \rangle_{I_\pm}, \langle \varphi_x^2 \rangle_{I_\pm})$  are also on the trajectory. Indeed, for the true Bellman function, each inequality in the Bellman induction of Section 6 is, in fact, an equality; thus, each split must be happening along the trajectory where the candidate is linear. Conversely, if  $m, n$  are two points on such a trajectory and we know the optimizers  $\varphi_m$  and  $\varphi_n$ , then the optimizer for any point  $x$  on the trajectory that is between  $m$  and  $n$  can be obtained simply by concatenating the two known optimizers:

$$\varphi_x(t) = \begin{cases} \varphi_m\left(\frac{t}{\gamma}\right), & t \in (0, \gamma), \\ \varphi_n\left(\frac{t-\gamma}{1-\gamma}\right), & t \in (\gamma, 1), \end{cases}$$

where  $\gamma = (x_1 - n_1)/(m_1 - n_1)$ .

*Remark 7.1.* In general, concatenating two functions from  $\text{BMO}_\varepsilon$  is not guaranteed to produce another  $\text{BMO}_\varepsilon$  function. One or both of the functions being concatenated,  $\varphi_n$  and  $\varphi_m$  in the formula above, may have to be rearranged to ensure the smallest possible BMO norm of the resulting function.

*Remark 7.2.* As we will see, the optimizers built in this section *do not depend* on the choice of the boundary function  $f$ , but only on the geometry of the canonical subdomain and, in some cases, on how such subdomains are glued together.

**7.1. Optimizers for  $L_0(x)$ .** The foliation of the domain  $\Omega_L(0)$  that corresponds to the Bellman candidate  $L_0$  given by (4.13) consists of horizontal lines. For each point  $x \in \Omega_L(0)$ , the extremal trajectory through  $x$  intersects the boundary of  $\Omega_\varepsilon$  in two points,  $(-\sqrt{x_2}, x_2)$  and  $(\sqrt{x_2}, x_2)$ . Since the only test functions available on the boundary of  $\Omega_\varepsilon$  are constants, we already know the optimizers for these points:  $\varphi_{(\pm\sqrt{x_2}, x_2)}(t) = \pm\sqrt{x_2}$ . Therefore, to construct the optimizer for the point  $x$ , we concatenate the two boundary values in the appropriate proportion:

$$(7.3) \quad \varphi_x(t) = \begin{cases} -\sqrt{x_2}, & t \in (0, \alpha), \\ \sqrt{x_2}, & t \in (\alpha, 1), \end{cases}$$

where

$$(7.4) \quad \alpha = \frac{1}{2} \left( 1 - \frac{x_1}{\sqrt{x_2}} \right).$$

We now verify (7.1) for  $\varphi_x$ .

**Lemma 7.3.** *The function  $\varphi_x$  given by (7.3) and (7.4) is an optimizer for  $L_0(x)$ .*

*Proof.* For part (a) of (7.1), we have  $\langle \varphi_x \rangle_{(0,1)} = -\alpha\sqrt{x_2} + (1-\alpha)\sqrt{x_2} = x_1$  and  $\langle \varphi_x^2 \rangle_{(0,1)} = \alpha x_2 + (1-\alpha)x_2 = x_2$ .

To show (b), we need to show that for any subinterval  $I$  of  $(0, 1)$  the Bellman point  $x^I$ , corresponding to  $I$  and  $\varphi_x$ , is in  $\Omega_\varepsilon$ . Observe that  $x^I$  is a convex combination of two boundary points,  $(\pm\sqrt{x_2}, x_2)$ . Thus, it lies on the line segment connecting these points and, furthermore, belongs to the convex subset  $\Omega_L(0)$  of  $\Omega_\varepsilon$ .

For (c), we trivially have  $\langle f(\varphi_x) \rangle_{(0,1)} = f(\sqrt{x_2}) = L_0(x)$ .  $\square$

**7.2. Optimizers for  $L^\pm(x; a)$ ,  $a \neq 0$ .** Recall that for each choice of  $a$  the candidates  $L^\pm(x; a)$ , given by (4.17) and (4.18), are built on the domain  $\Omega_L(a)$  consisting of the (closed) portion of  $\Omega_\varepsilon$  lying under a two-sided tangent  $x_2 = 2ax_1 + \varepsilon^2 - a^2$ . The Bellman foliation for  $L^+$  is the collection of straight lines connecting the corner point  $(u_+, u_+^2)$  to points  $(u, u^2)$  with  $u_- < u < u_+$ , while the foliation for  $L^-$  consists of lines connecting  $(u_-, u_-^2)$  to  $(u, u^2)$ . For each  $x \in \Omega_L(a)$ , except the two corners, there is a unique such  $u$ , given by (4.18):

$$(7.5) \quad u = \frac{x_2 - vx_1}{x_1 - v},$$

where  $v$  stands for either  $u_+$  or  $u_-$ , as appropriate.

Again, we already know the optimizers for the points  $(v, v^2)$  and  $(u, u^2)$ :  $\varphi_{(v, v^2)}(t) = v$  and  $\varphi_{(u, u^2)}(t) = u$ , respectively. Therefore, to construct the optimizer for the point  $x$  lying on the line connecting  $v$  and  $u$ , we concatenate these two optimizers:

$$(7.6) \quad \varphi_x(t) = \begin{cases} u, & t \in (0, \alpha); \\ v, & t \in (\alpha, 1), \end{cases}$$

where

$$(7.7) \quad \alpha = \frac{x_1 - v}{u - v} = \frac{(x_1 - v)^2}{x_2 - 2vx_1 + v^2}.$$

We now verify (7.1) for  $\varphi_x$ .

**Lemma 7.4.** *The function  $\varphi_x$  given by (7.5), (7.6), and (7.7) is an optimizer for  $L^-(x; a)$ , if  $v = u_-$ , and for  $L^+(x; a)$ , if  $v = u_+$ .*

*Proof.* For (a), we have  $\langle \varphi_x \rangle_{(0,1)} = \alpha u + (1 - \alpha)v = x_1$  and  $\langle \varphi_x^2 \rangle_{(0,1)} = \alpha u^2 + (1 - \alpha)v^2 = x_2$ .

For (b), we proceed as before. Let  $x^I$  be the Bellman point corresponding to  $\varphi_x$  and a subinterval  $I$  of  $(0, 1)$ . This point is a convex combination of the points  $(v, v^2)$  and  $(u, u^2)$ , which lie in the convex subset  $\Omega_L(a)$  of  $\Omega_\varepsilon$ . Therefore,  $x^I$  belongs to  $\Omega_\varepsilon$ .

Finally, we check the optimality relation (c) of (7.1):

$$\langle f(\varphi) \rangle_{(0,1)} = \alpha f(u) + (1 - \alpha)f(v) = \frac{f(v) - f(u)}{v - u} (x_1 - v) + f(v) = L(x; a),$$

according to (4.17). Here  $L$  stands for either  $L^+$  or  $L^-$ , as appropriate.  $\square$

**7.3. Optimizers for  $L(x; u_1 - \varepsilon) \rightarrow F^+(x; u_1, u_2)$ .** Recall formula (5.1) for  $F^+$  in this context:

$$F^+(x; u_1, u_2) = \frac{1}{\varepsilon} e^{-u/\varepsilon} \left[ \frac{f(u_1) - f(u_1 - 2\varepsilon)}{2} e^{u_1/\varepsilon} + \int_{u_1}^u f'(s) e^{s/\varepsilon} ds \right] (x_1 - u) + f(u),$$

with  $u$  given by (4.7):

$$u = u_+ = x_1 + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

The extremal trajectories for  $F^+$  are one-sided tangents to the upper boundary. According to the discussion in the introduction to this section, to be able to construct an optimizer for any  $x \in \Omega_{F^+}(u_1, u_2)$ , we first construct one for each point of the upper boundary of this sub-domain,  $\{(a, a^2 + \varepsilon^2), a \in (a_1, a_2)\}$ ,  $a_i = u_i - \varepsilon$ . Then, we concatenate the optimizer on the upper boundary with the constant optimizer on the bottom boundary via the extremal tangent through  $x$ .

Fix an  $a \in (a_1, a_2]$  and consider the tangent  $x_2 = 2ax_1 + \varepsilon^2 - a^2$  intersecting the lower boundary at the point  $(u, u^2)$ . Since all our extremal tangents are one-sided, we do not seem to have another optimizer with which to concatenate the constant function  $u$ . However, we circumvent this difficulty with the following approximating procedure. Fix a small number

$\Delta$  and let  $\alpha = \frac{\varepsilon - \Delta}{\varepsilon + \Delta}$ ,  $\beta = \frac{\varepsilon}{\varepsilon + \Delta}$ . Now consider the point  $(a - \Delta, a^2 + \varepsilon^2 - 2a\Delta)$ , also on the tangent. Assume for a moment that we know the optimizer  $\rho$  for this point and set

$$(7.8) \quad \psi_a(t) \approx \begin{cases} \rho\left(\frac{t}{\beta}\right), & t \in (0, \beta), \\ u, & t \in (\beta, 1). \end{cases}$$

To get the optimizer  $\rho$ , we draw a tangent through the point  $(a - \Delta, a^2 + \varepsilon^2 - 2a\Delta)$ , which intersects the upper boundary at the point  $(a - 2\Delta, (a - 2\Delta)^2 + \varepsilon^2)$  and the lower boundary, at the point  $(u - 2\Delta, (u - 2\Delta)^2)$  (see Figure 10). If we knew the optimizer  $\psi_{a-2\Delta}$ , we could again concatenate the optimizers on the two boundaries:

$$(7.9) \quad \rho(t) = \begin{cases} \psi_{a-2\Delta}\left(\frac{\varepsilon}{\varepsilon - \Delta}t\right), & t \in (0, 1 - \Delta/\varepsilon), \\ u - 2\Delta, & t \in (1 - \Delta/\varepsilon, 1). \end{cases}$$

Combining (7.8) and (7.9), we obtain a recursive approximation for  $\psi_a$ :

$$\psi_a(t) \approx \begin{cases} \psi_{a-2\Delta}\left(\frac{t}{\alpha}\right), & t \in (0, \alpha), \\ u - 2\Delta, & t \in (\alpha, \beta), \\ u, & t \in (\beta, 1). \end{cases}$$

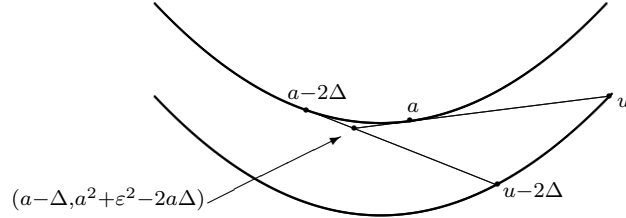


FIGURE 10. The construction of  $\psi_a$

Repeating this procedure  $k$  times, we get

$$(7.10) \quad \psi_a(t) \approx \psi_a^{(k)}(t) \stackrel{\text{def}}{=} \begin{cases} \psi_{a-2k\Delta}\left(\frac{t}{\alpha^k}\right), & t \in (0, \alpha^k), \\ u - 2k\Delta, & t \in (\alpha^k, \alpha^{k-1}\beta), \\ u - 2(k-1)\Delta, & t \in (\alpha^{k-1}\beta, \alpha^{k-2}\beta), \\ \dots & \dots \\ \dots & \dots \\ u - 2\Delta, & t \in (\alpha\beta, \beta), \\ u, & t \in (\beta, 1). \end{cases}$$

We seem to have a major problem: we do not know  $\psi_{a-2k\Delta}$ . We do, however, know  $\psi_{a_1}$ . Setting  $\Delta = \frac{a-a_1}{2k} = \frac{u-u_1}{2k}$ , we obtain a known function on the top line of (7.10). Now it is time to let  $k \rightarrow \infty$  (i.e.  $\Delta \rightarrow 0$ ). To determine  $\psi_a = \lim_{k \rightarrow \infty} \psi_a^{(k)}$ , we write down a simple differential equation: take a  $j$ ,  $1 < j < k$ , and let  $t = \alpha^j\beta$  be a generic point in  $(0, 1)$ . Then

$$\psi_a(t) - \psi_a(\alpha t) \approx \psi_a^{(k)}(t) - \psi_a^{(k)}(\alpha t) \approx (u - 2j\Delta) - (u - 2(j+1)\Delta) = 2\Delta.$$

On the other hand,

$$\psi_a(t) - \psi_a(\alpha t) \approx \psi_a'(t)t(1 - \alpha) \approx \psi_a'(t)t \frac{2\Delta}{\varepsilon}.$$

Combining the two approximate equalities and solving the differential equation, we obtain

$$\psi_a(t) = D + \varepsilon \log t.$$

Observe that

$$\lim_{k \rightarrow \infty} \alpha^k = \lim_{\Delta \rightarrow 0} \left( 1 - \frac{2\Delta}{\varepsilon + \Delta} \right)^{(a-a_1)/(2\Delta)} = e^{(a_1-a)/\varepsilon}.$$

Altogether, (7.10) becomes

$$(7.11) \quad \psi_a(t) = \begin{cases} \psi_{a_1}(e^{(a-a_1)/\varepsilon}t), & t \in (0, e^{(a_1-a)/\varepsilon}), \\ D + \varepsilon \log t, & t \in (e^{(a_1-a)/\varepsilon}, 1). \end{cases}$$

How do we determine the constant  $D$ ? Recall that we *a priori* have  $\langle \psi_{a_1} \rangle_{(0,1)} = a_1$  and must ensure  $\langle \psi_a \rangle_{(0,1)} = a$ . This gives

$$a_1 e^{(a_1-a)/\varepsilon} + (D - \varepsilon) \left( 1 - e^{(a_1-a)/\varepsilon} \right) + e^{(a_1-a)/\varepsilon} (a - a_1) = a$$

or

$$D = a + \varepsilon = u.$$

Having constructed an optimizer for each point on the upper boundary, we are in a position to construct one for any point  $x \in \Omega_{F^+}(u_1, u_2)$ . As planned, we consider the extremal tangent through  $x$ , and concatenate the optimizer  $\psi_a$  for the upper boundary and the constant  $u$  for the lower boundary. Specifically, we have

$$\varphi_x(t) = \begin{cases} \psi_a\left(\frac{\varepsilon}{u-x_1}t\right), & t \in (0, \frac{u-x_1}{\varepsilon}), \\ u, & t \in (\frac{u-x_1}{\varepsilon}, 1). \end{cases}$$

Using (7.11), we obtain the complete expression for the optimizer:

$$(7.12) \quad \varphi_x(t) = \begin{cases} \psi_{a_1}\left(\frac{t}{\mu\nu}\right), & t \in (0, \mu\nu), \\ u + \varepsilon \log \frac{t}{\mu}, & t \in (\mu\nu, \mu), \\ u, & t \in (\mu, 1), \end{cases}$$

where

$$(7.13) \quad \mu = \frac{u - x_1}{\varepsilon}, \quad \nu = e^{(u_1-u)/\varepsilon}, \quad \text{and} \quad u = u_+ = x_1 + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

Let us now recall the expression for  $\psi_{a_1}$ , the optimizer for  $L(a_1, a_1^2 + \varepsilon^2; a_1)$ . It is given by either (7.3) or (7.6), in each case with  $\alpha = 1/2$ :

$$\psi_{a_1}(t) = \begin{cases} a_1 - \varepsilon & \\ a_1 + \varepsilon & \end{cases} = \begin{cases} u_1 - 2\varepsilon, & t \in (0, 1/2), \\ u_1, & t \in (1/2, 1). \end{cases}$$

Therefore, formula (7.12) can be rewritten as

$$(7.14) \quad \varphi_x(t) = \begin{cases} u_1 - 2\varepsilon, & t \in (0, \mu\nu/2), \\ u_1, & t \in (\mu\nu/2, \mu\nu), \\ u + \varepsilon \log \frac{t}{\mu}, & t \in (\mu\nu, \mu), \\ u, & t \in (\mu, 1). \end{cases}$$

We can now prove

**Lemma 7.5.** *The function  $\varphi_x$  given by (7.13) and (7.14) is an optimizer for  $L(x; u_1 - \varepsilon) \longrightarrow F^+(x; u_1, u_2)$ .*



*Proof.* First observe that

$$\int_{\mu\nu}^{\mu} \log \frac{t}{\mu} dt = -\mu\nu \log \nu - \mu + \mu\nu, \quad \int_{\mu\nu}^{\mu} \log^2 \frac{t}{\mu} dt = -\mu\nu \log^2 \nu + 2\mu\nu \log \nu + 2(\mu - \mu\nu).$$

Now, using (7.13) and the fact that  $\langle \psi_{a_1} \rangle_{(0,1)} = a_1$ ,  $\langle \psi_{a_1}^2 \rangle_{(0,1)} = a_1^2 + \varepsilon^2$ , we get

$$\begin{aligned} \langle \varphi_x \rangle_{(0,1)} &= \frac{\mu\nu}{2}(u_1 - 2\varepsilon) + \frac{\mu\nu}{2}u_1 + u(1 - \mu\nu) + \varepsilon(-\mu\nu \log \nu - \mu + \mu\nu) \\ &= \mu\nu(u_1 - \varepsilon) + u(1 - \mu\nu) - \mu\nu(u_1 - u) - (u - x_1) + \varepsilon\mu\nu \\ &= x_1 \end{aligned}$$

and

$$\begin{aligned} \langle \varphi_x^2 \rangle_{(0,1)} &= \frac{\mu\nu}{2}(u_1 - 2\varepsilon)^2 + \frac{\mu\nu}{2}u_1^2 + u^2(1 - \mu\nu) + 2u\varepsilon(-\mu\nu \log \nu - \mu + \mu\nu) \\ &\quad + \varepsilon^2(-\mu\nu \log^2 \nu + 2\mu\nu \log \nu + 2\mu - 2\mu\nu) \\ &= x_2. \end{aligned}$$

Proving that  $\varphi_x \in \text{BMO}_\varepsilon$  is more delicate. Let  $q(t) = u + \varepsilon \log(t/\mu)$ . Observe that  $q \in \text{BMO}_\varepsilon((0, \infty))$ . Indeed, for  $l(t) \stackrel{\text{def}}{=} \log t$  and any interval  $(c, d) \subset (0, \infty)$  a direct calculation yields

$$(7.15) \quad \langle l^2 \rangle_{(c,d)} - \langle l \rangle_{(c,d)}^2 = 1 - \frac{cd}{(d-c)^2} \log^2 \left( \frac{d}{c} \right) \leq 1,$$

and so  $l \in \text{BMO}_1((0, \infty))$ , immediately implying the result for  $q$ . Moreover, setting  $c = 0$  in (7.15) shows that the Bellman point corresponding to  $q$  and any interval of the form  $(0, \gamma)$  is always on the upper boundary of  $\Omega_\varepsilon$ . The last bit of information we will need about  $q$ , one that can be shown by a computation similar to that at the beginning of the proof, is that

$$(7.16) \quad \langle q \rangle_{(0, \mu\nu)} = a_1, \quad \langle q^2 \rangle_{(0, \mu\nu)} = a_1^2 + \varepsilon^2.$$

Next, since  $\varphi_x$  is the cut-off at height  $u$  of the function

$$\eta_x(t) \stackrel{\text{def}}{=} \begin{cases} u_1 - 2\varepsilon, & t \in (0, \mu\nu/2), \\ u_1, & t \in (\mu\nu/2, \mu\nu), \\ q(t), & t \in (\mu\nu, 1), \end{cases}$$

it suffices to show that  $\eta_x \in \text{BMO}_\varepsilon$ , as Lemma 6.3 will then imply the result for  $\varphi_x$ . Thus, we aim to show that  $\delta_{c,d} \stackrel{\text{def}}{=} \langle \eta_x^2 \rangle_{(c,d)} - \langle \eta_x \rangle_{(c,d)}^2 \leq \varepsilon^2$  for all  $(c, d) \subset (0, 1)$ .

We note that the only subintervals  $(c, d)$  that need to be considered in detail are those with  $c \in (0, \mu\nu/2)$  and  $d \in (\mu\nu, 1)$ . Indeed, if  $0 \leq c < d \leq \mu\nu$ , then  $\delta_{c,d} \leq \varepsilon^2$ , since  $\psi_{a_1} \in \text{BMO}_\varepsilon$ , as has already been shown. If  $\mu\nu/2 \leq c < d \leq 1$ , then, again,  $\delta_{c,d} \leq \varepsilon^2$ , since  $\eta_x|_{(\mu\nu/2, 1)}$  is the cut-off, at height  $u_1$ , of the  $\text{BMO}_\varepsilon((\mu\nu/2, 1))$  function  $q$  and so Lemma 6.3 again applies.

Thus we focus on the case  $0 < c < \mu\nu/2 < \mu\nu < d < 1$ . Let  $z^- = (\langle \eta_x \rangle_{(0,c)}, \langle \eta_x^2 \rangle_{(0,c)})$ ,  $z = (\langle \eta_x \rangle_{(0,d)}, \langle \eta_x^2 \rangle_{(0,d)})$ , and  $z^+ = (\langle \eta_x \rangle_{(c,d)}, \langle \eta_x^2 \rangle_{(c,d)})$  be the three Bellman points corresponding to the intervals  $(0, c)$ ,  $(0, d)$ , and  $(c, d)$ , respectively. Since  $\eta_x|_{(0,c)}$  is constant,  $z^-$  is on the lower boundary of  $\Omega_\varepsilon$ . To locate  $z$ , we note that by (7.16) we have

$$(\langle q \rangle_{(0, \mu\nu)}, \langle q^2 \rangle_{(0, \mu\nu)}) = (\langle \eta_x \rangle_{(0, \mu\nu)}, \langle \eta_x^2 \rangle_{(0, \mu\nu)}),$$

and, therefore,

$$(\langle q \rangle_{(0,d)}, \langle q^2 \rangle_{(0,d)}) = (\langle \eta_x \rangle_{(0,d)}, \langle \eta_x^2 \rangle_{(0,d)}),$$

for all  $d \geq \mu\nu$ . As noted above,  $z$  must be on the upper boundary of  $\Omega_\varepsilon$ .

Consider now the line through  $z^-$  and  $z$ . Since  $z$  is, by construction, to the right of the point  $(a_1, a_1^2 + \varepsilon^2)$ , this line lies above the top boundary of  $\Omega_L(a_1)$  and so first exits  $\Omega_\varepsilon$  and then re-enters it at  $z$ . Since  $z$  is a convex combination of  $z^-$  and  $z^+$ ,  $z^+$  is located on the same line, to the right of  $z$ , and, therefore, inside  $\Omega_\varepsilon$  (see Figure 11).

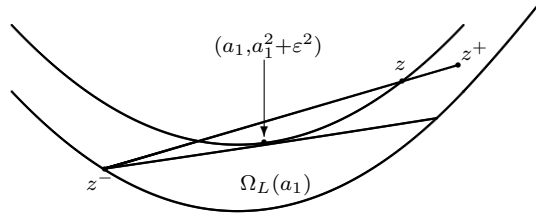


FIGURE 11. The mutual location of  $z^-$ ,  $z$ , and  $z^+$

For part (c) of (7.1), we have

$$\begin{aligned}
 \langle f(\varphi_x) \rangle_{(0,1)} &= \frac{\mu\nu}{2} f(u_1 - 2\varepsilon) + \frac{\mu\nu}{2} f(u_1) + \int_{\mu\nu}^{\mu} f(u + \varepsilon \log(t/\mu)) dt + f(u)(1 - \mu) \\
 &= \frac{\mu\nu}{2} f(u_1 - 2\varepsilon) + \frac{\mu\nu}{2} f(u_1) + \frac{\mu}{\varepsilon} \int_{u_1}^u e^{(s-u)/\varepsilon} f(s) ds + f(u)(1 - \mu) \\
 &= \frac{1}{2} e^{(u_1-u)/\varepsilon} \mu [f(u_1 - 2\varepsilon) - f(u_1)] - \mu \int_{u_1}^u e^{(s-u)/\varepsilon} f'(s) ds + f(u) \\
 &= F^+(x; u_1, u_2).
 \end{aligned}$$

Here we have used the formula (7.14) for  $\varphi_x$  on the first step, the change of variables  $s = u + \varepsilon \log(t/\mu)$  on the second, integration by parts on the third, and expressions (7.13) for  $\mu$  and  $\nu$  on the third and fourth. This completes the proof of the lemma.  $\square$

**7.4. Optimizers for  $F^+(x; -\infty, u_2)$ .** This case follows directly from the previous one. Here the candidate  $F^+$  is given by

$$F^+(x; -\infty, u_2) = \frac{1}{\varepsilon} e^{-u/\varepsilon} \left[ \int_{-\infty}^u f'(s) e^{s/\varepsilon} ds \right] (x_1 - u) + f(u),$$

with  $u$  given, as before, by

$$u = u_+ = x_1 + \varepsilon - \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

We readily obtain an optimizer for this case by setting  $\nu = 0$  in (7.14):

$$(7.17) \quad \varphi_x(t) = \begin{cases} u + \varepsilon \log \frac{t}{\mu}, & t \in (0, \mu), \\ u, & t \in (\mu, 1). \end{cases}$$

Lemma 7.5 can be restated for this case:

**Lemma 7.6.** *The function  $\varphi_x$  given by (7.17) is an optimizer for  $F^+(x; -\infty, u_2)$ .*

**7.5. Optimizers for sequences containing  $F^-$ .** To construct optimizers for  $F^-$  we use the symmetry  $F^-(x_1, x_2; u_1, u_2) = F^+(-x_1, x_2; -u_2, -u_1)$ . Therefore, in the optimizers given

by (7.14) and (7.17) we have to replace  $\varphi$  by  $-\varphi$  and  $u_1$  by  $-u_2$ . This yields

$$(7.18) \quad \varphi_x(t) = \begin{cases} u_2 + 2\varepsilon, & t \in (0, \mu\nu/2), \\ u_2, & t \in (\mu\nu/2, \mu\nu), \\ u - \varepsilon \log \frac{t}{\mu}, & t \in (\mu\nu, \mu), \\ u, & t \in (\mu, 1), \end{cases}$$

for  $F^-(x; u_1, u_2) \leftarrow L(x; u_2 + \varepsilon)$  and

$$(7.19) \quad \varphi_x(t) = \begin{cases} u - \varepsilon \log \frac{t}{\mu}, & t \in (0, \mu), \\ u, & t \in (\mu, 1), \end{cases}$$

for  $F^-(x; u_1, \infty)$ . Here

$$(7.20) \quad \mu = \frac{x_1 - u}{\varepsilon}, \quad \nu = e^{(u - u_2)/\varepsilon}, \quad \text{and} \quad u = u_- = x_1 - \varepsilon + \sqrt{\varepsilon^2 - x_2 + x_1^2}.$$

Lemma 7.5 and Lemma 7.6 can be reformulated, respectively, as

**Lemma 7.7.** *The function  $\varphi_x$  given by (7.18) and (7.20) is an optimizer for  $F^-(x; u_1, u_2) \leftarrow L(x; u_2 + \varepsilon)$ .*

**Lemma 7.8.** *The function  $\varphi_x$  given by (7.19) and (7.20) is an optimizer for  $F^-(x; u_1, \infty)$ .*

**7.6. Optimizers for sequences containing  $T(x; u)$ .** Constructing optimizers for a  $T$ -type candidate is straightforward, since  $T(x; u)$  is a linear function in  $\Omega_T(u)$ . Here is our strategy: for each point  $x \in \Omega_T(u)$ , draw any straight line that intersects both straight-line sides of  $\Omega_T(u)$ , but not the upper boundary of  $\Omega_\varepsilon$ , and then concatenate the optimizers for the two points of intersection in the appropriate proportion. Since  $T$  is linear, the resulting function will be an optimizer for it. This requires knowing optimizers along both bounding tangents, which is consistent with the fact that  $T$  is both left- and right-incomplete. Thus, we will eventually need to examine the specific sequences in which  $T$  shows up in our global Bellman candidates, in order to write down an explicit optimizer for each case. However, most of the construction, as well as the verification of parts (a) and (c) of (7.1), can be carried out without specifying the left and right neighbors of  $T$ .

Suppose  $T(x; u)$  is glued to a Bellman candidate  $G^-(x)$  along its left bounding tangent and a candidate  $G^+$  along the right one. Let us assume that we know optimizers for  $G^\pm$  along their respective bounding tangents.

For any  $x \in \Omega_T(u)$  there is always a way to draw a line through  $x$  intersecting the left and right tangents at the points  $x^-$  and  $x^+$ , respectively, and such that the whole segment  $[x^-, x^+]$  is in  $\Omega_T(u)$ . (For example, at least one of the tangents to the upper boundary of  $\Omega_\varepsilon$  that pass through  $x$  will satisfy this requirement. In fact, for each point of  $\Omega_T(u)$  other than the three corners both tangents will satisfy it.) Then, we can concatenate the two known optimizers,  $\varphi_{x^-}$  and  $\varphi_{x^+}$ , to obtain a test function  $\varphi_x$  for the point  $x$ :

$$(7.21) \quad \varphi_x(t) = \begin{cases} \varphi_{x^-}\left(\frac{t}{\beta}\right), & t \in (0, \beta), \\ \varphi_{x^+}\left(\frac{t - \beta}{1 - \beta}\right), & t \in (\beta, 1), \end{cases}$$

where

$$(7.22) \quad \beta = \frac{x_1^+ - x_1}{x_1^+ - x_1^-} = \frac{x_2^+ - x_2}{x_2^+ - x_2^-}.$$

We can verify parts (a) and (c) of (7.1) directly from (7.21) and (7.22). For the averages, we have

$$\langle \varphi_x \rangle_{(0,1)} = x_1^- \beta + x_1^+ (1 - \beta) = x_1, \quad \langle \varphi_x^2 \rangle_{(0,1)} = x_2^- \beta + x_2^+ (1 - \beta) = x_2.$$

To check the optimality of  $\varphi_x$ , we write  $T = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0$  and calculate

$$\begin{aligned} \langle f(\varphi) \rangle_{(0,1)} &= G^-(x^-)\beta + G^+(x^+)(1-\beta) \\ &= T(x^-; u)\beta + T(x^+; u)(1-\beta) \\ &= (\alpha_1 x_1^- + \alpha_2 x_2^- + \alpha_0)\beta + (\alpha_1 x_1^+ + \alpha_2 x_2^+ + \alpha_0)(1-\beta) \\ &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0 = T(x; u). \end{aligned}$$

Here we have used: on the first step, the optimality of  $\varphi_{x^-}$  and  $\varphi_{x^+}$  for  $G^-$  and  $G^+$ , respectively; on the second and third steps, the boundary conditions (4.22) for  $T$  along the two bounding tangents; and on the last step, the fact that  $x = \beta x^- + (1-\beta)x^+$ .

Before specifying  $\varphi_x$  for each sequence involving  $T(x; u)$ , let us rewrite (7.21) in the form that is independent of  $x^-$  and  $x^+$ . Since each bounding tangent is assumed to be an extremal trajectory for the corresponding Bellman candidate (the left tangent for  $G^-$ , the right one for  $G^+$ ), the candidate is linear along the tangent. Thus we can write  $\varphi_{x^-}$ , an optimizer for  $G^-(x^-)$ , as a concatenation of optimizers for  $G^-(u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2)$  and  $G^-(u, u^2)$  and similarly for  $\varphi_{x^+}$ :

$$(7.23) \quad \varphi_{x^-}(t) = \begin{cases} \psi_{u-\varepsilon}\left(\frac{t}{\alpha_-}\right), & t \in (0, \alpha_-), \\ u, & t \in (\alpha_-, 1), \end{cases} \quad \varphi_{x^+}(t) = \begin{cases} u, & t \in (0, \alpha_+), \\ \psi_{u+\varepsilon}\left(\frac{t-\alpha_+}{1-\alpha_+}\right), & t \in (\alpha_+, 1). \end{cases}$$

where

$$\alpha_- = \frac{u - x_1^-}{\varepsilon}, \quad \alpha_+ = 1 - \frac{x_1^+ - u}{\varepsilon}.$$

Observe that in the expression for  $\varphi_{x^-}$  we put the constant value  $u$  on the right part of  $(0, 1)$ , while in the expression for  $\varphi_{x^+}$  this constant value is on the left. This is done so as to minimize the BMO norm of the resulting function  $\varphi_x$  given by (7.21) (see Remark 7.1). Using (7.21) in conjunction with (7.23), we get

$$(7.24) \quad \varphi_x(t) = \begin{cases} \psi_{u-\varepsilon}\left(\frac{t}{\beta\alpha_-}\right), & t \in (0, \beta\alpha_-), \\ u, & t \in (\beta\alpha_-, \beta + (1-\beta)\alpha_+), \\ \psi_{u+\varepsilon}\left(\frac{t-\beta-(1-\beta)\alpha_+}{(1-\beta)(1-\alpha_+)}\right), & t \in (\beta + (1-\beta)\alpha_+, 1). \end{cases}$$

Since  $x$  is a unique — and independent of  $x^-$  and  $x^+$  — convex combination of the points  $(u - \varepsilon, u^2 - 2\varepsilon u + 2\varepsilon^2)$ ,  $(u, u^2)$ , and  $(u + \varepsilon, u^2 + 2\varepsilon u + 2\varepsilon^2)$ , we expect the weights  $\beta\alpha_-$  and  $(1-\beta)(1-\alpha_+)$  in (7.24) not to depend on how the points  $x^\pm$  were chosen. Indeed, after a bit of algebra, we can rewrite (7.24) as

$$(7.25) \quad \varphi_x(t) = \begin{cases} \psi_{u-\varepsilon}\left(\frac{t}{\mu_-}\right), & t \in (0, \mu_-), \\ u, & t \in (\mu_-, 1 - \mu_+), \\ \psi_{u+\varepsilon}\left(\frac{t-1+\mu_+}{\mu_+}\right), & t \in (1 - \mu_+, 1), \end{cases}$$

where

$$(7.26) \quad \mu_- = \frac{x_2 - 2ux_1 + u^2}{4\varepsilon^2} - \frac{x_1 - u}{2\varepsilon}, \quad \mu_+ = \frac{x_2 - 2ux_1 + u^2}{4\varepsilon^2} + \frac{x_1 - u}{2\varepsilon}.$$

We now need to check that  $\varphi_x$  constructed according to (7.25) will be in  $\text{BMO}_\varepsilon$  for each global context of  $T$ . Among our four canonical blocks, a  $T$  block can only have  $F^+$  or  $L$  blocks glued to its left and only  $F^-$  or, again,  $L$  blocks to its right. Since we have already constructed optimizers for all  $F^\pm$  and  $L$  blocks, we could proceed in generality and prove that an optimizer of the form (7.25) could always be rearranged — separately on  $(0, \mu_-)$  and  $(1 - \mu_+, 1)$  — so that the resulting function is in  $\text{BMO}_\varepsilon$ . However, we choose here to be more

explicit and consider specific optimizers for the specific sequences in which  $T$  appears. We have three such sequences and so split further presentation in three parts.

7.6.1. *Optimizers for  $F^+(x; -\infty, 0) \rightarrow T_0(x) \leftarrow F^-(x; 0, \infty)$ .* This sequence appears in (5.17). For this case (7.25) gives

$$(7.27) \quad \varphi_x(t) = \begin{cases} \varepsilon \log\left(\frac{t}{\mu_-}\right), & t \in (0, \mu_-), \\ 0, & t \in (\mu_-, 1 - \mu_+), \\ -\varepsilon \log\left(\frac{t - 1 + \mu_+}{\mu_+}\right), & t \in (1 - \mu_+, 1). \end{cases}$$

To show that  $\varphi_x \in \text{BMO}_\varepsilon$ , take an interval  $(c, d) \subset (0, 1)$ . First, observe that if  $c \in [\mu_-, 1 - \mu_+]$ , the Bellman point  $x^{(c,d)}$  is the same as the one for the cut-off at height 0 of the function  $\varepsilon \log(t/\mu_-)$ , which has been shown to be in  $\text{BMO}_\varepsilon$ ; therefore,  $x^{(c,d)}$  is in  $\Omega_\varepsilon$ . The same reasoning applies when  $d \in [\mu_-, 1 - \mu_+]$ . Therefore, we will assume that  $c \in (0, \mu_-)$  and  $d \in (1 - \mu_+, 1)$ .

Recall representation (7.22), which gives  $x$  as a convex combination of  $x^-$  and  $x^+$ . In our standard notation,  $x^- = x^{(0,\beta)}$  and  $x^+ = x^{(\beta,1)}$ . We first would like to determine the location of the point  $x^{(c,\beta)}$ . We know that  $x^{(0,c)}$  is the Bellman point for the interval  $(0, c)$  and the logarithm  $\varepsilon \log(t/\mu_-)$  and we have already seen that every such point is on the parabola  $x_2 = x_1^2 + \varepsilon^2$ . Therefore,  $x^{(0,c)}$  is above the line  $m$  through  $x^-$  and  $x^+$  (see Figure 12). Since

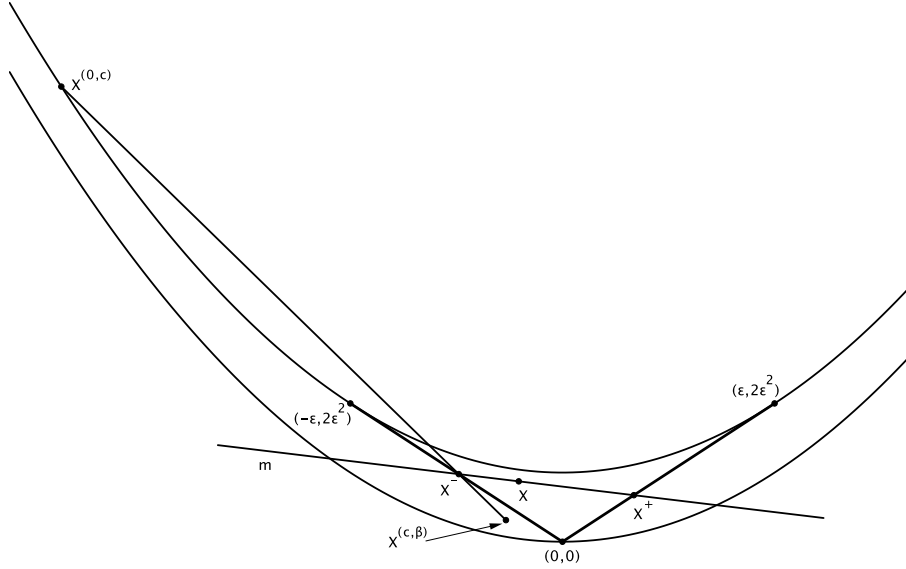


FIGURE 12. The location of the point  $x^{(c,\beta)}$

$x^-$  is a convex combination of  $x^{(0,c)}$  and  $x^{(c,\beta)}$ , we conclude that  $x^{(c,\beta)}$  is below  $m$ . Similarly, the point  $x^{(\beta,d)}$  (not shown in the figure) and, therefore, the whole line segment  $[x^{(c,\beta)}, x^{(\beta,d)}]$  is below  $m$  and so in  $\Omega_\varepsilon$ . The point  $x^{(c,d)}$  is a convex combination of  $x^{(c,\beta)}$  and  $x^{(\beta,d)}$ , which means that it is on the line segment and, thus, in  $\Omega_\varepsilon$ .

7.6.2. *Optimizers for  $L_0(x) \rightarrow F^+(x; \varepsilon, \xi) \rightarrow T(x; \xi) \leftarrow F^-(x; \xi, \infty)$ .* This sequence appears in (5.26). Note that we have to include in the sequence the block  $L_0$  to the left of  $F^+$ ,

since  $F^+$  is left-incomplete. For this case (7.25) yields

$$(7.28) \quad \varphi_x(t) = \begin{cases} -\varepsilon, & t \in (0, \mu_- \nu / 2), \\ \varepsilon, & t \in (\mu_- \nu / 2, \mu_- \nu), \\ \xi + \varepsilon \log\left(\frac{t}{\mu_-}\right), & t \in (\mu_- \nu, \mu_-), \\ \xi, & t \in (\mu_-, 1 - \mu_+), \\ \xi - \varepsilon \log\left(\frac{t-1+\mu_+}{\mu_+}\right), & t \in (1 - \mu_+, 1), \end{cases}$$

where  $\nu = e^{1-\xi/\varepsilon}$  and  $\mu_{\pm}$  are given by (7.26).

Instead of  $\varphi_x$ , it is convenient to consider  $\tilde{\varphi}_x = \varphi_x - \xi$ :

$$\tilde{\varphi}_x(t) = \begin{cases} -\xi - \varepsilon, & t \in (0, \mu_- \nu / 2), \\ -\xi + \varepsilon, & t \in (\mu_- \nu / 2, \mu_- \nu), \\ \varepsilon \log\left(\frac{t}{\mu_-}\right), & t \in (\mu_- \nu, \mu_-), \\ 0, & t \in (\mu_-, 1 - \mu_+), \\ -\varepsilon \log\left(\frac{t-1+\mu_+}{\mu_+}\right), & t \in (1 - \mu_+, 1). \end{cases}$$

Clearly,  $\varphi_x \in \text{BMO}_{\varepsilon} \iff \tilde{\varphi}_x \in \text{BMO}_{\varepsilon}$ . Let us take a subinterval  $(c, d)$  of  $(0, 1)$ . Then  $\tilde{\varphi}_x|_{(\mu_- \nu, 1)}$  is just (the appropriate restriction of) the optimizer from the previous subsection, given by (7.27), and so in  $\text{BMO}_{\varepsilon}$ . Furthermore,  $\tilde{\varphi}_x|_{(\mu_- \nu / 2, 1)}$  is the cut-off of that optimizer at height  $\varepsilon - \xi$ , restricted to  $(\mu_- \nu / 2, 1)$ , and hence, it is also  $\text{BMO}_{\varepsilon}$ . Therefore, in proving that the Bellman point  $x^{(c, d)}$  for  $\tilde{\varphi}_x$  is in  $\Omega_{\varepsilon}$  we only need to consider  $0 < c < \mu_- \nu / 2$ . Where should we place  $d$ ? If  $d \leq 1 - \mu_+$ ,  $x^{(c, d)}$  can be seen to be a Bellman point for the optimizer for the sequence  $L \rightarrow F^+$ , given by (7.14) with  $u_1 = -\xi + \varepsilon$  and  $u = 0$ ; therefore, it is in  $\Omega_{\varepsilon}$ . Thus, from now on we will assume that  $c < \mu_- \nu / 2 < 1 - \mu_+ < d$ .

As before, we appeal to representation (7.21)–(7.22). The only difference now is that the point  $x^{(0, c)} = (-\xi - \varepsilon, (\xi + \varepsilon)^2)$  will not be on the upper parabola  $x_2 = x_1^2 + \varepsilon^2$ , but instead on the lower parabola  $x_2 = x_1^2$ . Since  $\xi > \varepsilon$ , this point is above the line  $x_2 = -2\varepsilon x_1$ . Therefore, it is above the line through  $x^-$  and  $x^+$  (let us again call it  $m$ ). Since  $x^- = x^{(0, \beta)}$  is a convex combination of  $x^{(0, c)}$  and  $x^{(c, \beta)}$ , we conclude that  $x^{(c, \beta)}$  is below  $m$ . From here the consideration is identical to the one in the previous subsection and we conclude that  $x^{(c, d)} \in \Omega_{\varepsilon}$ .

**7.6.3. Optimizers for  $L^+(x; -\varepsilon) \rightarrow T_0(x) \leftarrow L^-(x; \varepsilon)$ .** This sequence appears in (5.29). From (7.25) we obtain

$$(7.29) \quad \varphi_x(t) = \begin{cases} -2\varepsilon, & t \in (0, \mu_- / 2), \\ 0, & t \in (\mu_- / 2, 1 - \mu_+ / 2), \\ 2\varepsilon, & t \in (1 - \mu_+ / 2, 1). \end{cases}$$

To prove that  $\varphi_x \in \text{BMO}_{\varepsilon}$ , take an interval  $(c, d) \subset (0, 1)$ . If  $c \geq \mu_- / 2$  and/or  $d \leq 1 - \mu_+ / 2$ , the Bellman point  $x^{(c, d)}$  is a convex combination of one of  $(\pm 2\varepsilon, 4\varepsilon^2)$  and  $(0, 0)$ . The point  $x^{(c, d)}$  then is on one of the tangent lines  $x_2 = \pm 2\varepsilon x_1$  and, thus, in  $\Omega_{\varepsilon}$ . Let us assume that  $c < \mu_- / 2$  and  $1 - \mu_+ / 2 < d$ . More fully, let us write  $c < \mu_- / 2 < \beta < 1 - \mu_+ / 2 < d$ , where  $\beta$  is given by (7.22). We have  $x^- = x^{(0, \beta)}$  as a convex combination of  $x^{(0, c)} = (-2\varepsilon, 4\varepsilon^2)$  and  $x^{(c, \beta)}$ . Therefore,  $x^{(c, \beta)}$  is on the tangent  $x_2 = -2\varepsilon x_1$ , below  $x^-$ . Similarly,  $x^{(\beta, d)}$  is below  $x^+$ . We conclude that  $x^{(c, d)} \in \Omega_{\varepsilon}$ .

We have completed the proofs of the following sequence of lemmas.

**Lemma 7.9.** *The function (7.27), with  $\mu_{\pm}$  given by (7.26) for  $u = 0$ , is an optimizer for  $F^+(x; -\infty, 0) \rightarrow T_0(x) \leftarrow F^-(x; 0, \infty)$ .*

**Lemma 7.10.** *The function (7.28), with  $\nu = e^{1-\xi/\varepsilon}$  and  $\mu_{\pm}$  given by (7.26) for  $u = \xi$ , is an optimizer for  $L_0(x) \longrightarrow F^+(x; \varepsilon, \xi) \longrightarrow T(x; \xi) \longleftarrow F^-(x; \xi, \infty)$ .*

**Lemma 7.11.** *The function (7.29), with  $\mu_{\pm}$  given by (7.26) for  $u = 0$ , is an optimizer for  $L^+(x; -\varepsilon) \longrightarrow T_0(x) \longleftarrow L^-(x; \varepsilon)$ .*

Putting together Lemmas 7.3–7.11, we obtain

**Theorem 7.12.** *For any  $x \in \Omega_{\varepsilon}$ , we have*

$$\begin{aligned} \text{For } p \geq 2 : \quad & N_{\varepsilon,p}(x) \leq \mathbf{B}_{\varepsilon,p}(x), \quad M_{\varepsilon,p}(x) \geq \mathbf{b}_{\varepsilon,p}(x), \\ \text{For } 1 \leq p < 2 : \quad & M_{\varepsilon,p}(x) \leq \mathbf{B}_{\varepsilon,p}(x), \quad N_{\varepsilon,p}(x) \geq \mathbf{b}_{\varepsilon,p}(x), \\ \text{For } 0 < p < 1 : \quad & P_{\varepsilon,p}(x) \leq \mathbf{B}_{\varepsilon,p}(x), \quad R_{\varepsilon,p}(x) \geq \mathbf{b}_{\varepsilon,p}(x). \end{aligned}$$

## 8. PROOFS OF THE MAIN INEQUALITIES

Theorems 6.5 and 7.12 give us the explicit expressions for  $\mathbf{B}_{\varepsilon,p}(x)$  and  $\mathbf{b}_{\varepsilon,p}(x)$  for all  $p > 0$ :

**Theorem 8.1.** *For any  $x \in \Omega_{\varepsilon}$ , we have*

$$\begin{aligned} \text{For } p \geq 2 : \quad & \mathbf{B}_{\varepsilon,p}(x) = N_{\varepsilon,p}(x), \quad \mathbf{b}_{\varepsilon,p}(x) = M_{\varepsilon,p}(x), \\ \text{For } 1 \leq p < 2 : \quad & \mathbf{B}_{\varepsilon,p}(x) = M_{\varepsilon,p}(x), \quad \mathbf{b}_{\varepsilon,p}(x) = N_{\varepsilon,p}(x), \\ \text{For } 0 < p < 1 : \quad & \mathbf{B}_{\varepsilon,p}(x) = P_{\varepsilon,p}(x), \quad \mathbf{b}_{\varepsilon,p}(x) = R_{\varepsilon,p}(x). \end{aligned}$$

We are now in a position to prove all the theorems stated in Section 2.

*Proof of Theorem 2.1.* It suffices to consider  $Q = (0, 1)$ . Take any  $\varphi \in \text{BMO}(Q)$  and let  $\varepsilon = \|\varphi\|_{\text{BMO}(Q)}$ ,  $x_1 = \langle \varphi \rangle_Q$ ,  $x_2 = \langle \varphi^2 \rangle_Q$ . Then

$$\mathbf{b}_{\varepsilon,p}(x_1, x_2) \leq \langle |\varphi|^p \rangle_Q \leq \mathbf{B}_{\varepsilon,p}(x_1, x_2).$$

Replacing  $\varphi$  with  $\varphi - \langle \varphi \rangle_Q$  gives

$$(8.1) \quad \mathbf{b}_{\varepsilon,p}(0, x_2 - x_1^2) \leq \langle |\varphi - \langle \varphi \rangle_Q|^p \rangle_Q \leq \mathbf{B}_{\varepsilon,p}(0, x_2 - x_1^2).$$

We now invoke Theorem 8.1, for which we need the exact expressions for candidates  $M, N, P$ , and  $R$ . They come, respectively, from (5.15), (5.18), (5.27), and (5.30):

$$M_{\varepsilon,p}(0, x_2) = x_2^{p/2}; \quad N_{\varepsilon,p}(0, x_2) = \frac{p}{2} \Gamma(p) \varepsilon^{p-2} x_2; \quad P_{\varepsilon,p}(0, x_2) = x_2^{p/2}; \quad R_{\varepsilon,p}(0, x_2) = 2^{p-2} \varepsilon^{p-2} x_2.$$

Plugging these into (8.1) yields the stated inequalities. Furthermore, these inequalities are sharp because each becomes an equality, if we take  $\varphi$  to be the corresponding optimizer  $\varphi_{(0,\varepsilon^2)}$  from Section 7. Specifically, let

$$(8.2) \quad \varphi_1 = \begin{cases} -\varepsilon; & t \in (0, \frac{1}{2}), \\ \varepsilon; & t \in (\frac{1}{2}, 1), \end{cases} \quad \varphi_2 = \begin{cases} \varepsilon \log(4t), & t \in (0, \frac{1}{4}), \\ 0, & t \in (\frac{1}{4}, \frac{3}{4}), \\ -\varepsilon \log(4-4t), & t \in (\frac{3}{4}, 1), \end{cases} \quad \varphi_3 = \begin{cases} -2\varepsilon, & t \in (0, \frac{1}{8}), \\ 0, & t \in (\frac{1}{8}, \frac{7}{8}), \\ 2\varepsilon, & t \in (\frac{7}{8}, 1). \end{cases}$$

These are the optimizers given by (7.3), (7.27), and (7.29), respectively, each constructed for the point  $(0, \varepsilon^2)$ . As was shown in Section 7,  $\|\varphi_k\|_{\text{BMO}(Q)} = (\langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2)^{1/2} = \varepsilon$ ,  $k = 1, 2, 3$ . On the other hand,

$$\langle |\varphi_1 - \langle \varphi_1 \rangle_Q|^p \rangle_Q = \langle |\varphi_1|^p \rangle_Q = \varepsilon^p,$$

$$\langle |\varphi_2 - \langle \varphi_2 \rangle_Q|^p \rangle_Q = \langle |\varphi_2|^p \rangle_Q = 2\varepsilon^p \cdot \frac{1}{4} \int_0^1 |\log t|^p dt = \frac{1}{2} \Gamma(p+1) \varepsilon^p,$$

and

$$\langle |\varphi_3 - \langle \varphi_3 \rangle_Q|^p \rangle_Q = \langle |\varphi_3|^p \rangle_Q = 2 \cdot \frac{1}{8} \cdot (2\varepsilon)^p = 2^{p-2} \varepsilon^p.$$

□

*Proof of Theorem 2.2.* Again, set  $Q = (0, 1)$ . Take any  $\varphi \in \text{BMO}(Q)$  and let  $\varepsilon = \|\varphi\|_{\text{BMO}(Q)}$ . For any subinterval  $J$  of  $Q$  let  $x_1^J = \langle \varphi \rangle_J$ ,  $x_2^J = \langle \varphi^2 \rangle_J$ . Arguing as in the previous proof, we have

$$\mathbf{b}_{\varepsilon,p}(0, x_2^J - (x_1^J)^2) \leq \langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J \leq \mathbf{B}_{\varepsilon,p}(0, x_2^J - (x_1^J)^2).$$

Let us consider these inequalities separately. For the one on the right, using that  $\mathbf{B}_{\varepsilon,p}(0, \cdot)$  is increasing, we have

$$\langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J \leq \mathbf{B}_{\varepsilon,p}(0, \varepsilon^2).$$

Taking the supremum over all  $J$ , we get

$$\|\varphi\|_{\text{BMO}^p(Q)}^p \leq \mathbf{B}_{\varepsilon,p}(0, \varepsilon^2).$$

For each  $p$  this inequality is sharp: for  $p < 2$  it is attained for  $\varphi = \varphi_1$  from (8.2), and for  $p > 2$  it is attained for  $\varphi = \varphi_2$ .

For the inequality on the left, take a sequence  $\{J_n\}$  of subintervals of  $Q$  such that

$$\lim_{n \rightarrow \infty} (x_2^{J_n} - (x_1^{J_n})^2) = \varepsilon^2.$$

Using the continuity of  $\mathbf{b}_{\varepsilon,p}(0, \cdot)$ , we have

$$\mathbf{b}_{\varepsilon,p}(0, \varepsilon^2) = \lim_{n \rightarrow \infty} \mathbf{b}_{\varepsilon,p}(0, x_2^{J_n} - (x_1^{J_n})^2) \leq \limsup_{n \rightarrow \infty} \langle |\varphi - \langle \varphi \rangle_{J_n}|^p \rangle_{J_n} \leq \|\varphi\|_{\text{BMO}^p(Q)}^p.$$

This inequality is sharp for  $p > 2$ : it is attained, again, for  $\varphi = \varphi_1$  from (8.2).  $\square$

*Remark 8.2.* The last calculation in the proof shows why  $\varphi_2$  and  $\varphi_3$  cannot be used to show sharpness of the norm estimates in the cases  $1 \leq p < 2$  and  $0 < p \leq 1$ , respectively. Consider  $p = 1$ . The issue is that, while each function attains its  $\text{BMO}^2$  norm on  $(0, 1)$ , neither attains its  $\text{BMO}^1$  norm on this interval. Indeed, one can easily calculate that both functions have 1-oscillations equal to  $\varepsilon/2$  on  $(0, 1)$ . However,

$$\|\varphi_2\|_{\text{BMO}^1} \geq \langle |\varphi_2 - \langle \varphi_2 \rangle_{(0,1/4)}| \rangle_{(0,1/4)} = \frac{2\varepsilon}{e}, \quad \|\varphi_3\|_{\text{BMO}^1} \geq \langle |\varphi_3 - \langle \varphi_3 \rangle_{(0,1/4)}| \rangle_{(0,1/4)} = \varepsilon.$$

*Proof of Theorem 2.4.* We can set  $Q = (0, 1)$ . If  $\|\varphi\|_{\text{BMO}(Q)} = 0$ , there is nothing to prove. Assuming this is not the case, we use Theorem 2.1 to get

$$\frac{2\|\varphi\|_{\text{BMO}(Q)}^{2-p_2}}{p_2\Gamma(p_2)} \langle |\varphi - \langle \varphi \rangle_Q|^{p_2} \rangle_Q \leq \langle \varphi^2 \rangle_Q - \langle \varphi \rangle_Q^2 \leq \frac{2\|\varphi\|_{\text{BMO}(Q)}^{2-p_1}}{p_1\Gamma(p_1)} \langle |\varphi - \langle \varphi \rangle_Q|^{p_1} \rangle_Q.$$

Each of these inequalities becomes an equality when  $\varphi = \varphi_2$  from (8.2). The left-hand side inequality in the statement of the theorem is attained, for instance, for  $\varphi = \varphi_1$ .  $\square$

*Proof of Theorem 2.5.* Set  $Q = (0, 1)$ ; all averages will be over  $Q$ . Let  $\varepsilon = \|\varphi\|_{\text{BMO}(Q)}$  and assume  $\varepsilon < 1$ . We have

$$\begin{aligned} \langle e^{|\varphi - \langle \varphi \rangle|} \rangle &= \sum_{k=0}^{\infty} \frac{1}{k!} \langle |\varphi - \langle \varphi \rangle|^k \rangle \leq 1 + \mathbf{B}_{\varepsilon,1}(0, \varepsilon^2) + \sum_{k=2}^{\infty} \frac{1}{k!} \mathbf{B}_{\varepsilon,k}(0, \varepsilon^2) \\ &= 1 + M_{\varepsilon,1}(0, \varepsilon^2) + \sum_{k=2}^{\infty} \frac{1}{k!} N_{\varepsilon,k}(0, \varepsilon^2) = 1 + \varepsilon + \frac{1}{2} \sum_{k=2}^{\infty} \varepsilon^k = \frac{1 - \frac{\varepsilon^2}{2}}{1 - \varepsilon}. \end{aligned}$$

On the other hand, taking  $\varphi = \varphi_2$  from (8.2), we get

$$\langle e^{|\varphi_2 - \langle \varphi_2 \rangle|} \rangle = 2 \cdot \frac{1}{4} \int_0^1 e^{\varepsilon |\log t|} dt + 2 \cdot \frac{1}{4} = \frac{1 - \frac{\varepsilon}{2}}{1 - \varepsilon}.$$

This calculation also shows that the bound  $\varepsilon_0 = 1$  is sharp.  $\square$



9. OTHER CHOICES OF  $f$ 

Throughout the paper, we have concentrated on one specific boundary function  $f : f(s) = |s|^p$ ,  $p > 0$ . However, the machinery developed in these pages works for many other choices of  $f$ . Let us briefly describe several such choices and their Bellman functions without going into details.

9.1.  $f(s) = \log |s|$ . As mentioned earlier, this function corresponds to the case  $p = 0$ , since

$$\lim_{p \rightarrow 0} \langle |\varphi|^p \rangle^{1/p} = e^{\langle \log |\varphi| \rangle}.$$

It is easy to show that the corresponding Bellman functions are

$$(9.1) \quad \mathbf{B}_{\varepsilon,0}(x) = P_{\varepsilon,0}(x) \quad \text{and} \quad \mathbf{b}_{\varepsilon,0}(x) = -\infty.$$

Here  $P_{\varepsilon,0}$  is given by (5.26), with every block re-specified for  $f(s) = \log |s|$ . Thus,  $F^+$  and  $F^-$  are given by (5.1) and (5.7), respectively;  $T$  is given by (4.27); and  $L_0$ , given by (4.13), is simply  $\frac{1}{2} \log x_2$ . To show that this is a viable global candidate, one needs Lemma 5.1. To prove the statement for  $\mathbf{B}$ , use the local concavity of the candidate  $P_{\varepsilon,0}$  to run the induction of Section 6 and then apply the optimizer for  $P_{\varepsilon,p}$  from Section 7. To prove the statement for  $\mathbf{b}$ , simply use the optimizer for  $R_{\varepsilon,p}$  from Section 7.

From (9.1), we have sharp inequalities for  $\varphi \in \text{BMO}_\varepsilon$ :

$$(9.2) \quad -\infty \leq \langle \log |\varphi| \rangle \leq P_{\varepsilon,0}(\langle \varphi \rangle, \langle \varphi \rangle^2),$$

and so

$$(9.3) \quad 0 \leq e^{\langle \log |\varphi| - \langle \varphi \rangle \rangle} \leq e^{P_{\varepsilon,0}(0, \langle \varphi \rangle^2) - \langle \varphi \rangle^2} = e^{\frac{1}{2} \log(\langle \varphi \rangle^2 - \langle \varphi \rangle^2)} \leq \varepsilon.$$

While the second inequality in (9.2) is non-trivial and gives a sharp estimate on  $\langle \log |\varphi| \rangle$  for any pair of specified averages of  $\varphi$ , its immediate consequence, the second-from-left inequality in (9.3), is not interesting, as it simply expresses the norm monotonicity (1.1). However, the leftmost inequality in (9.3) is important, as its sharpness means that  $\text{BMO} \subsetneq \text{BMO}^0$ . It is also the same result as one gets from the top line in Theorem 2.2, by taking the limit as  $p \rightarrow 0^+$ .

9.2.  $f(s) = |s|^p$ ,  $p < 0$ . For this case, the situation reverses, compared to the previous one. We have

$$\mathbf{B}_{\varepsilon,p}(x) = \infty, \quad \mathbf{b}_{\varepsilon,p}(x) = P_{\varepsilon,p}(x),$$

with  $P_{\varepsilon,p}$  given by (5.26) and so (5.27), and so

$$P_{\varepsilon,p}(\langle \varphi \rangle, \langle \varphi \rangle^2) \leq \langle |\varphi|^p \rangle \leq \infty,$$

which produces the sharp inequalities

$$0 \leq \langle |\varphi| - \langle \varphi \rangle \rangle^{1/p} \leq [P_{\varepsilon,p}(0, \langle \varphi \rangle^2 - \langle \varphi \rangle^2)]^{1/p} = (\langle \varphi \rangle^2 - \langle \varphi \rangle^2)^{1/2} \leq \varepsilon.$$

9.3.  $f(s) = e^{|s|} - |s|$ ,  $\varepsilon < 1$ . This is the function that implicitly allowed us to prove the John–Nirenberg estimates of Theorem 2.5. As we saw in Section 8, the key fact is that all upper Bellman functions  $\mathbf{B}_{\varepsilon,p}$  and, separately, all lower Bellman functions  $\mathbf{b}_{\varepsilon,p}$  have identical optimizers for  $p \geq 2$ . Since the Taylor expansion for  $e^{|s|} - |s|$  is missing the term corresponding to  $p = 1$ , we expect the Bellman foliation of  $\Omega_\varepsilon$  to be the same as for  $|s|^p$ ,  $p \geq 2$ . Indeed, we can easily show that

$$(9.4) \quad \mathbf{B}_{\varepsilon,f}(x) = N_{\varepsilon,f}(x), \quad \mathbf{b}_{\varepsilon,f}(x) = M_{\varepsilon,f}(x),$$

where  $M_{\varepsilon,f}$  and  $N_{\varepsilon,f}$  are given by (5.14) and (5.17), with their blocks re-specified for this choice of  $f$ . Therefore, after a small bit of calculation, we have the sharp inequalities

$$e^{\sqrt{\langle \varphi^2 \rangle - \langle \varphi \rangle^2}} - \sqrt{\langle \varphi^2 \rangle - \langle \varphi \rangle^2} \leq \langle e^{|\varphi| - \langle \varphi \rangle} \rangle - \langle |\varphi| - \langle \varphi \rangle \rangle \leq \frac{\langle \varphi^2 \rangle - \langle \varphi \rangle^2}{2(1 - \varepsilon)} + 1 \leq \frac{1 - \varepsilon + \frac{\varepsilon^2}{2}}{1 - \varepsilon}.$$

In fact, if  $f$  is any linear combination of powers greater than or equal to 2, with non-negative coefficients, the Bellman functions will be given by (9.4). If, on the other hand,  $f$  is a linear combination of powers between 1 and 2 with non-negative coefficients, the upper and lower Bellman functions will switch, i.e. they will be given by  $M_{\varepsilon,f}$  and  $N_{\varepsilon,f}$ , respectively. Further generalizations along these lines are possible.

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